FREENESS FOR TENSORS

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ANATOMY OF A TENSOR

$$\boxed{T = \left(T_{i_1 \dots i_p}\right)_{1 \leq i_\ell \leq N_\ell}}$$

p = order

$T_{i_1 \dots i_p} \in \mathbb{R}$ or $\mathbb{C}$

ℓ = a leg

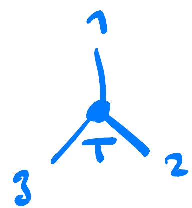
N_ℓ = dimension of the leg $= N$ (in this talk)

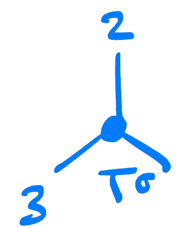
$p=1$: vector, $p=2$: matrix , $p \geq 3$: hypermatrix

PERMUTATION OF LEGS

$$T = \left(T_{i_1,\dots,i_p}\right)_{1 \le i_\ell \le N}$$

for $\sigma \in S_p$ $\qquad T^\sigma = \left(T_{i_{\sigma(1)},\dots,i_{\sigma(p)}}\right)$

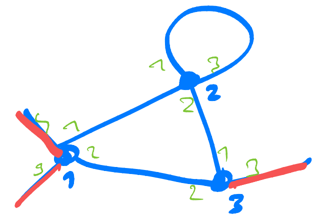




$\sigma = (1\,2)$

If $T_i \in \mathbb{R}$ and $T^\sigma = T$ $\forall \sigma$, T is a <u>symmetric tensor</u>

TENSOR MAPS



m = map with boundary $\in \mathcal{M}_3$

$\tau = (T_v)_{v \in V(m)}$ a collection of tensors

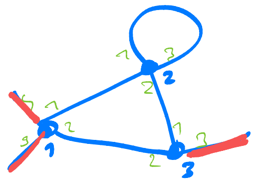
deg(v) = order of T_v

$\gamma = \begin{cases} 0 & \text{if boundary} \\ \text{c.c.} & \text{otherwise} \end{cases}$

tensor of order $|\partial|$:

$$\boxed{m(\tau)_{i_\partial} = \frac{1}{N^\gamma} \sum_{i \in [N]^{E(m)}} \prod_{v \in V(m)} T_{i_{\partial v}}}$$

TENSOR MAPS

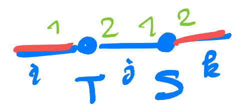


$$m(\tau)_i = \frac{1}{N^{\gamma}} \sum_{i \in [N]^{E(m)}} \prod_{v \in V(m)} T_{i_{\partial v}}$$

matrix trace :

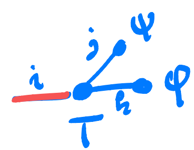
$$m(x) = \frac{1}{N} \sum_i T_{ii}$$

matrix multiplication :



$$m(x)_{ik} = \sum_j T_{ij} S_{jk}$$

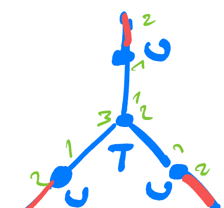
contraction by vectors



$$m(x) = \sum_{j,k} T_{ijk} \psi_j \varphi_k$$

unitary composition :

$T.U^p =$



$$(T.U)_i = \sum_j T_j \prod_{\ell=1}^{p} U_{j_\ell i_\ell}$$

...

TRACE MAPS

trace maps : $m \in \mathcal{M}_0$ (no boundary).

$$m(\mathcal{T}) \in \mathbb{C}$$

$$\forall U \in \mathcal{U}_N \quad \boxed{m(T_1 \cdot U^{p_1}, T_2 \cdot U^{p_2}, \dots, T_n \cdot U^{p_n}) = m(T_1, \dots, T_n)}$$

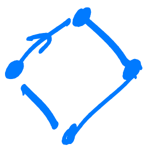
→ trace maps are the building block of a spectral theory for tensors.

GURAU'S SPECTRAL MEASURE

Assume T of order $p \geq 2$, T symmetric real

$$I_n(T) := \sum m(T,\dots,T)$$

all p-regular unlabeled rooted connected maps with n vertices



Thm (Gurau) We have: $I_n(T) = \int \lambda^n \mu_T(\lambda)$ for all $n \geq 0$ for some μ_T probability measure on $\mathbb{R}$.

$p=2$: $\mu_T = \mathrm{ESD}$ ⚠ for $p \geq 3$ μ_T has unbounded support.

DISTRIBUTION OF TENSORS

Let $\mathcal{A} = \{a_i, i \in I\}$ a finite collection of tensors

Def The distribution of $\mathcal{A}$ is the collection

$$(m(T)) \; : \; \underset{\in \mathcal{M}_0}{m} \text{ map}, \quad T = (T_v)_{v \in V(m)}$$

$T_v \in \mathcal{A}$.

consistent order.

Ex. If $\mathcal{A} = \{a\}$, $a \in M_N(\mathbb{R})$

$$\frac{1}{N} \mathrm{Tr}\left(a^{\varepsilon_1} \cdots a^{\varepsilon_k}\right) \qquad a^{\varepsilon_i} \in \{a, a^T\}.$$

WIGNER RANDOM TENSOR

let $(X_i)_{i=(i_1,\dots,i_p)/S_p}$ be iid real random variables

$\mathbb{E}(X_i) = 0$, $\mathbb{E}(X_i^2) = \frac{1}{(p-1)!}$, $\mathbb{E}|X_i|^k \leq C_k \quad \forall k$

Thm (Gurau, Bonnin) $W^N = \frac{(X_i)_{i\in[N]^p}}{N^{\frac{p-1}{2}}}$ converges in distribution (in probability)

$\forall m \in \mathcal{M}_0,\ m(W^N) \xrightarrow[N\to+\infty]{} \mu(m)$

ASYMPTOTIC FREENESS

Let A_1^N, A_2^N be two collections of tensors of dimension N.

1. we want to compute as $N \to +\infty$

$m(T)$, $T = (T_v)_v$, $T_v \in A_1^N \cup A_2^N$, from the dist. of A_ε^N, $\varepsilon = 1,2$

when A_1^N and A_2^N are in "generic" position

$\Rightarrow$ For matrices: Voiculescu's asymptotic freeness, aka deterministic equivalent

2. We want to prove that natural model of random tensors are asymptotically free

TENSOR FREENESS

MAP ACTION ON TENSOR SPACES

let $\mathcal{E}_0 = \mathbb{C}$, $\mathcal{E}_1 =$ vector space $\mathcal{E}_p = \mathcal{E}_1^{\otimes p}$.

$$\mathcal{E} = \bigcup \mathcal{E}_p$$

We assume that a map $m \in \mathcal{M}_q$ with q boundary edge acts on $\mathcal{E}_m = \bigtimes_{v \in V(m)} \mathcal{E}_{\deg(v)}$

$$m : \begin{array}{c} x = (x_v)_{v \in V(m)} \longmapsto m(x) \\ \mathcal{E}_m \longrightarrow \mathcal{E}_q. \end{array}$$

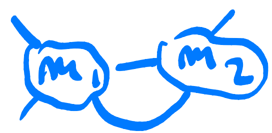
ex: $\mathcal{E}_1 = \mathbb{C}^N$, $m(x) =$ tensor map

$\mathcal{E}_2 = \mathbb{C}^N \otimes \mathbb{C}^N \simeq M_N(\mathbb{C})$

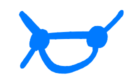
MAP ACTION ON TENSOR SPACES

This action satisfies:

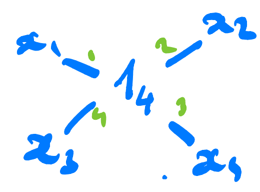
- Linear
- Class invariance: $m(x) = m'(x)$ if $m \sim m'$
- Morphism: $\big((m_1)\,(m_2)\big)(x) = m_1(x) \otimes m_2(x)$
- Substitution:

  if $m =$ 

  $m(x) = m_0(m_1(x), m_2(x))$

  m_0 

- Identity for even p:



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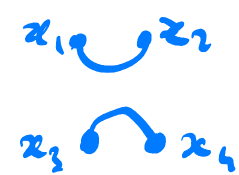


MAP VECTOR BUNDLE

Def $\langle \mathcal{M} \rangle$ - Bundle:

Let $A \subseteq \mathcal{E} = \bigcup_p \mathcal{E}_p$.

$\langle A \rangle$ is the union of vector spaces spanned

by $m(x)$, $m \in \bigcup_{q \geq 0} \mathcal{M}_q$, $x_i \in A \cup 1_{2p}$.

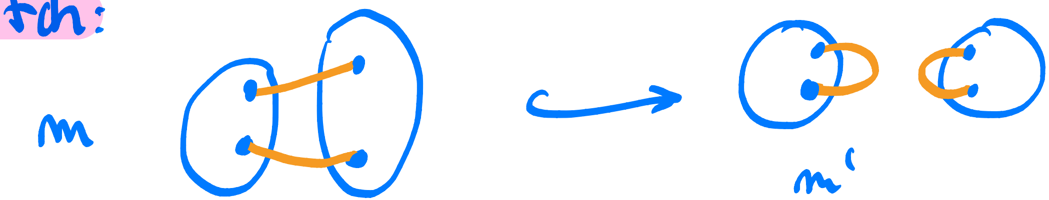
For matrices, we recover the algebra generated by St.

NON-CROSSING POSET OF MAPS

$\mathcal{S}_\pi$: the set of maps $m \in \mathcal{M}_0$ with n vertices and given degree sequence:

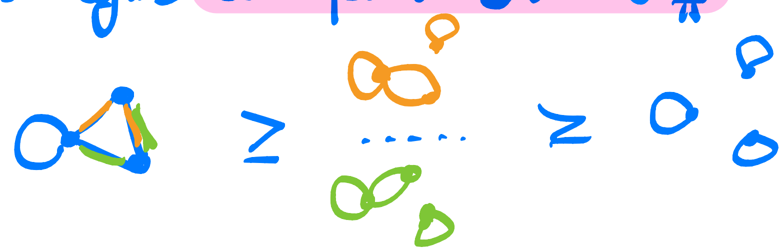
$$\pi = (\pi_1, \dots, \pi_n)$$

NC switch:



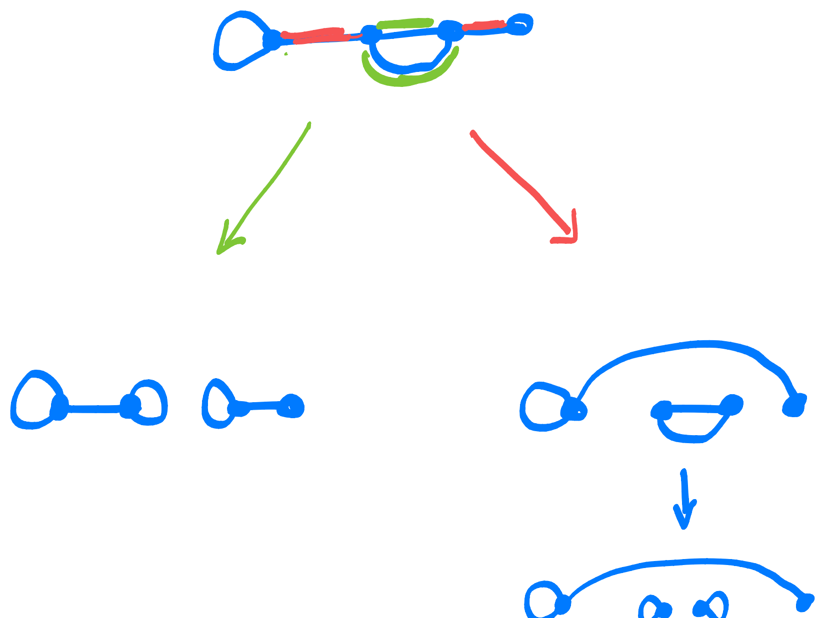
nb of c.c. of m = nb of c.c. of m' $-$ 1

This defines a poset on $\mathcal{S}_\pi$



NON-CROSSING POSET OF MAPS

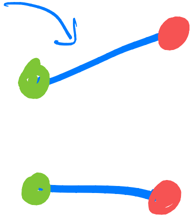
⚠ For odd degrees, there could be multiple minimal elts!



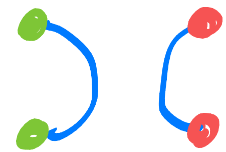
DEFINITION OF FREENESS

let A, B be two $\mathcal{M}$-bundles.

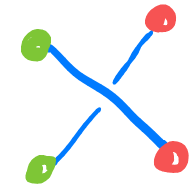
chromatic edge



chromatic NC switch



or



type + type −

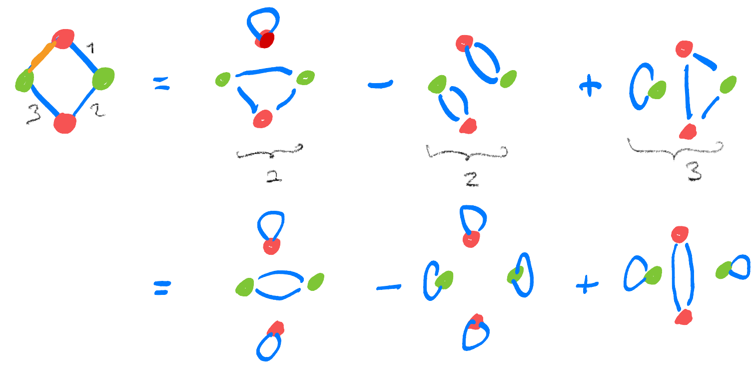
Def A and B are free if for any $(\overset{\text{no } \varepsilon}{m}, x)$ and chromatic edge e

$$x_v \in A \cup B \qquad m(x) = \sum_f m^+_{ef}(x) - \sum_f m^-_{ef}(x)$$

sums over chromatic edges $f \neq e$ s.t. $m^{+/-}_{ef} < m$

DEFINITION OF FREENESS

For ex.



* Freeness characterizes the distribution of $A \cup B$

* If $m(\pi)$ is minimal and non-monochromatic then $m(\pi) = 0$

FREE CUMULANTS

There is an analog of Speicher's cumulant theorem in free probability

By Moebius inversion, we can define $K_\pi(x)$ by the formula:

$$\boxed{m(x) = \sum_{\pi \leq m} K_\pi(x)}$$

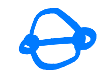
Thm Let A, B be even (= only even order tensors). A and B are free iff $\forall\, m$, $K_m(x) = 0$ if there exist $u \neq v$ s.t. $x_u \in A$, $x_v \in B$.

HIGH ORDER SEMI-CIRCULAR

Thm (Gurau, Bonnin) $W^N = \dfrac{(X_i)_{i\in[N]^p}}{N^{\frac{p-1}{2}}}$ converges in distribution (in probability)

$\forall m \in \mathcal{M}_0,\quad m(W^N) \xrightarrow[N\to+\infty]{} \mu(m)$

* We have $\mu(m) = m(s,\dots,s) = m(s)$

  and $K_m(s) = 0$ unless $m =$  "melon"

  $K_{\ominus}(s) = 1$

* $m(s) = \mathbb{1}_{\{m \text{ is melonic}\}}$ 

  $\#\{\text{melonic with } 2k \text{ vertices}\} = \text{Fuss-Catalan}(k,p) = \dfrac{1}{pk+1}\dbinom{pk+1}{k}$

ASYMPTOTIC FREENESS

ASYMPTOTIC FREENESS FOR WIGNER TENSORS

$$W_N = \frac{(X_i)_{i \in [N]^p}}{N^{\frac{p-1}{2}}}.$$

$A_1^N = \{a_i^N, \ i \in I\}$ a finite collection in $\mathcal{E}^N = \bigcup_p \mathbb{C}^{N^{\otimes p}}$
such that $\mathrm{order}(a_i^N) = p_i$
and $|m(x)| \leq C(m)$ $\forall N$ $\forall m$
si $x_v \in A_1^N$

Thm W_N and A_1^N are asymptotically free in probability.

ASYMPTOTIC FREENESS FOR HAAR UNITARIES

$M_N(\mathbb{C}) \ni U_N$: Haar distributed on $\mathcal{O}_N$ or $\mathcal{U}_N$

$\mathcal{A}_1^N = \{ a_i^N, \ i \in I \}$ a finite collection in $\mathcal{E}^N = \bigcup_p \mathbb{C}^{N^{\otimes p}}$ such that order $(a_i^N) = p_i$ and $|m(x)| \leq C(m)$ $\forall N$, $\forall m$, $x_v \in \mathcal{A}_1^N$

Thm $\{U_N, U_N^*\}$ and $\mathcal{A}_1^N$ are asymptotically free in probability

ASYMPTOTIC FREENESS FOR UNITARILY INV. FAMILIES

$\varepsilon \in \{1,2\}$ $\mathcal{A}^N_\varepsilon = \{ a^N_{i,\varepsilon} : i \in I_\varepsilon \}$ a finite collection in $\mathcal{E}^N = \bigcup_p \mathbb{C}^{N^{\otimes p}}$

such that order $(a^N_{i,\varepsilon}) = p_{i,\varepsilon}$

and $|m(x)| \leq C(m)$ $\forall N$ $\forall m$

$x_v \in \mathcal{A}^N_\varepsilon$

$\mathcal{A}_2 \cdot U_N^{\#} = \{ a^N_{i,2} \cdot U_N^{p_{i,2}} , i \in I_2 \}$

U_N : Haar distributed on O_N or U_N

Thm $\mathcal{A}^N_1$ and $\mathcal{A}_2 \cdot U_N^{\#}$ are asymptotically free in probability

SCHNINGER – DYSON LOOP EQUATIONS

The proofs are based on **recursion equations** satisfied at fixed dimension N for Gaussian or other unitary invariant models.

For Example:

If $W^N = \frac{(X_i)_{i\in[N]^p}}{N^{\frac{p-1}{2}}}$ is Gaussian

$\forall\, m, u, T = (T_v)_{v\in V(m)}$

$u \in V(m)$

$$\mathbb{E}[m(T)] = \frac{1}{(p-1)!}\sum_{v,\sigma} \mathbb{E}\left[(m^{u}.\sigma)^{\prime v}(T)\right] \times \frac{N^{c.c.(m^{u}.\sigma)^{\prime v}}}{N^{c.c.\,m+p-1}}$$

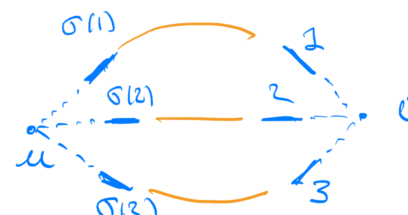
$T_v \in \{W^N\} \cup \mathcal{A}_0^N$

$T_u = W^N$

$v \in V(m)$

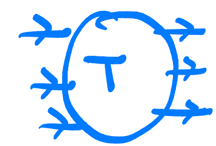
$\sigma \in S_p$

$T_v = W^N$

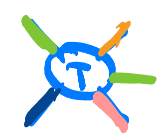


CONCLUDING WORDS

VARIANTS

* We may consider extra symmetries, for examples tensors have even order  with 'input/output legs'

Collins-Gurau-Lionni (2024), Nechita-Park (2024), Kunisky-Moore-Wein (2024)

* To consider tensors with legs of various dimensions, we may decorate the edges of maps. 

PERSPECTIVES

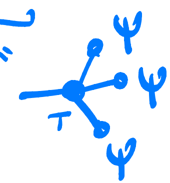
* Examples of computations of m? such as random hypergraphs

* Concentration inequalities for $m(T)$?

* Resolvent for tensors?

* Connections to Z-eigenvalues / eigenvectors?

$\psi \in \mathbb{R}^N$, $\|\psi\|_2 = 1$, $\qquad \underbrace{T\cdot\psi^{p-1}}_{=} = \lambda\psi$



THANK YOU FOR YOUR ATTENTION!