Canonical coordinates for moduli spaces of rank two irregular connections on curves (joint work with A. Komyo, F. Loray, M.-H. Saito, arXiv:2309.05012)

Szilárd Szabó

Budapest University of Technology and Economics

September 29th 2023
Équations différentielles motiviques et au–delà
Institut Henri Poincaré

Outline

Outline

Outline

Outline

Outline

Motivation

- Isomonodromy equations (Poincaré, Painlevé, Garnier, R. Fuchs, Schlesinger,..., Dubrovin, Jimbo–Miwa–Ueno, Okamoto, Iwasaki, Boalch,...)
- Geometry of moduli spaces of sheaves on Poisson surfaces and Hilbert schemes of points on symplectic surfaces (Mukai, Beauville, Donagi–Markman,...)
- Separation of variables in Hitchin systems (Sklyanin, Beauville–Narasimhan–Ramanan, Adams–Harnad–Hurtubise, Hurtubise, Gorsky–Nekrasov–Rubtsov, ...)
- Opers (N. Katz, Beilinson–Drinfeld,...)
- Confluence of singular points of connections (Gaiur–Mazzocco–Rubtsov, Klimeš,...)
- Mirror symmetry and cluster algebras (Kontsevich–Odesskii, Kontsevich–Soibelman, Gross–Hacking–Keel, Fock–Goncharov,...)

Notation

- $r = 2$
- $\mathfrak{h} \subset \mathfrak{gl}(2, \mathbb{C})$ standard Cartan subalgebra
- $\mathfrak{h}_0 \subset \mathfrak{h}$ its regular part
- $\theta^{\pm}$ eigenvalues of $\theta \in \mathfrak{h}$
- $I = \{1, \ldots, \nu\}$ for some $\nu \in \mathbb{Z}_+$

Irregular curve (Boalch) with residues

Fixed data

- C smooth projective curve of genus g
- $D = \sum_{i\in I} m_i[t_i]$ an effective divisor on C ($m_i \in \mathbb{Z}_+$, $t_i \neq t_j$ for $i \neq j$)
- z_i a local coordinate centered at t_i
- $\{\boldsymbol{\theta}_i\}_{i\in I}$ where $\boldsymbol{\theta}_i = (\theta_{i,-m_i}, (\theta_{i,-m_i+1}, \ldots, \theta_{i,-2})) \in \mathfrak{h}_0 \times \mathfrak{h}^{m_i-2}$
- $\boldsymbol{\theta}_{\mathrm{res}} = (\theta_{1,-1}, \theta_{2,-1}, \ldots, \theta_{\nu,-1})$ where $\theta_{i,-1} \in \mathfrak{h}$

Assumptions

- $n = \deg D = \sum_{i\in I} m_i$ satisfies $4g - 3 + n > 0$
- $\sum_{i=1}^{\nu} \mathrm{tr}(\theta_{i,-1}) = -(2g-1)$
- $\sum_{i=1}^{\nu} \theta_{i,-1}^{\pm} \notin \mathbb{Z}$
- for all $i \in I$ such that $m_i = 1$ the eigenvalues of $\theta_{i,-1}$ do not differ by an integer

Meromorphic connection over irregular curve with residues

- $E \to C$ a holomorphic rank 2 vector bundle of degree $2g - 1$
- $\nabla \colon E \to E \otimes \Omega^1_C(D)$ meromorphic connection (necessarily irreducible!)
- such that in some trivialization of $E|_{m_i[t_i]}$ we have

$$\nabla = \mathrm{d} + \theta_{i,-m_i} \frac{\mathrm{d}z_i}{z_i^{m_i}} + \theta_{i,-m_i+1} \frac{\mathrm{d}z_i}{z_i^{m_i-1}} + \cdots + \theta_{i,-2} \frac{\mathrm{d}z_i}{z_i^2} + \theta_{i,-1} \frac{\mathrm{d}z_i}{z_i}$$

- M_{dR} moduli space of meromorphic connections over fixed irregular curve with residues

Cyclic vector, apparent singularities

- Riemann–Roch $\Rightarrow$ for generic E we have $\dim_{\mathbb{C}} H^0(C, E) = 1$.
- cyclic vector: a generator $\mathbf{e}_1 \in H^0(C, E)$
- $E_0 \subset E$ rank 2 locally free subsheaf generated by $\mathbf{e}_1, \nabla_\partial(\mathbf{e}_1)$ for all $\partial \in T_C(-D) = (\Omega^1_C(D))^{-1}$
- splitting $E_0 \cong \mathcal{O}_C \oplus (\Omega^1_C(D))^{-1}$
- $\phi_\nabla \colon E_0 \longrightarrow E$ inclusion
- $\nabla_0 = \phi_\nabla^*(\nabla) \colon E_0 \to E_0 \otimes \Omega^1_C(D + B)$
- B apparent singularities of ∇
- $N := \deg(B) = 4g - 3 + n = \frac{1}{2} \dim_{\mathbb{C}} M_{\mathrm{dR}}$

Assumptions

- B is reduced
- $\mathrm{Supp}(B) \cap \mathrm{Supp}(D) = \varnothing$
- $B = q_1 + \cdots + q_N$.

Companion normal form

- With respect to the frame $(\mathbf{e}_1, \nabla_0(\mathbf{e}_1))$ of E_0 we have

$$\nabla_0 = \begin{pmatrix} \mathrm{d} & \beta \\ 1 & \delta \end{pmatrix}$$

- $\mathrm{d}\colon \mathcal{O}_C \to \Omega^1_C$ trivial connection
- δ a connection in $(\Omega^1_C(D))^{-1}$ with polar divisor $D + B$
- $\beta \in (\Omega^1_C(D))^{\otimes 2} \otimes \mathcal{O}_C(B)$
- $1\colon \mathcal{O}_C \to (\Omega^1_C(D))^{-1} \otimes \Omega^1_C(D) \cong \mathcal{O}_C$ identity

Properties of the connection δ

- Polar part of δ over D: determined by the irregular curve with residues
- Polar part of δ over B: logarithmic with residue $+1$
- δ is determined by the irregular curve with residues up to $H^0(C, \Omega^1_C)$
- choice of $\delta \rightsquigarrow g$ free parameters

Properties of the quadratic differential β

- Laurent series at t_i

$$\beta = \left(\beta_{i,-2m_i} z_i^{-2m_i} + \cdots + \beta_{i,-2} z_i^{-2} + O(z_i^{-1})\right) (\mathrm{d}z_i)^{\otimes 2}.$$

- $\beta_{i,-2m_i}, \ldots, \beta_{i,-2}$ are uniquely determined by the irregular curve with residues
- Laurent series at q_j

$$\beta = \left(\beta_{j,-2} z_j^{-2} + \beta_{j,-1} z_j^{-1} + O(z_j^0)\right) (\mathrm{d}z_j)^{\otimes 2}.$$

- $\beta_{j,-2} = 0$
- set $\zeta_j \, \mathrm{d}z_j = \mathrm{res}_{q_j}(\beta) \in \Omega^1_C(D)|_{q_j}$
- summarizing:

$$\mathrm{res}_{q_j} \nabla_0 = \begin{pmatrix} 0 & \zeta_j \, \mathrm{d}z_j \\ 0 & 1 \end{pmatrix}.$$

- geometric interpretation: quasi-parabolic structure of E_0 over B, different from $\mathcal{O}_C \subset E_0$

Generic independence

- For fixed $\{(q_j, \zeta_j \, \mathrm{d}z_j)\}_{j=1}^N$, β is determined by the irregular curve with residues up to $H^0(C, (\Omega_C^1)^{\otimes 2}(D))$
- choice of $\beta \rightsquigarrow 3g - 3 + n$ free parameters
- recall: g free parameters for δ
- $\deg(B) = 4g - 3 + n = N$
- condition: q_j are apparent singularities

Set $\mathbf{\Omega}(D) =$ total space of $\Omega_C^1(D)$.

Proposition

For generic data $\{(q_j, \zeta_j \, \mathrm{d}z_j)\}_{j=1}^N \in \mathrm{Sym}^N(\mathbf{\Omega}(D))$ there exist unique β and δ as above such that ∇_0 has apparent singularities at all the points q_j $(1 \leq j \leq N)$, and such that $\mathrm{res}_{q_j}(\beta) = \zeta_j \, \mathrm{d}z_j$.

Generic independence: sketch of proof

- Condition for q_j to be apparent:

$$(\beta - \zeta_j \delta \otimes \mathrm{d}z_j - \zeta_j^2 \, \mathrm{d}z_j^{\otimes 2})(q_j) = 0.$$

- $(\omega_l)_{l=1}^{g}$, $(\nu_k)_{k=1}^{N-g}$ bases of $H^0(C, \Omega^1_C)$ and $H^0(C, (\Omega^1_C)^{\otimes 2}(D))$ respectively
- fix any (δ_0, β_0) with apparent singularities $q_1 + \cdots + q_N$
- take base expansions

$$\begin{cases} \beta &= \beta_0 + b_1 \nu_1 + \cdots + b_{N-g} \nu_{N-g} \\ \delta &= \delta_0 + d_1 \omega_1 + \cdots + d_g \omega_g \end{cases}$$

- linear system of N equations in N variables b_k, d_l
- for generic choices the determinant does not vanish
- for $g > 0$ there always exist special choices such that the determinant vanishes

Affine bundle

- Let $c_d = c_1(E) \in H^2(C, \mathbb{C}) \cong \mathrm{Ext}^1_{\mathcal{O}_C}(T_C, \mathcal{O}_C)$
- Consider the corresponding locally free rank 2 extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{A}_C(c_d) \longrightarrow T_C \longrightarrow 0$$

- It gives rise to the Atiyah–Lie algebroid

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{A}_C(c_d, D) \longrightarrow T_C(-D) \longrightarrow 0$$

- affine bundle $\Omega^1_C(D, c_d)$ modelled on $\Omega^1_C(D)$:

$$\Omega^1_C(D, c_d) = \left\{ \phi \in \mathcal{A}_C(c_d, D)^\vee \;\middle|\; \langle \phi, 1_{\mathcal{A}_C(c_d,D)} \rangle = 1 \right\}.$$

- total space of $\Omega^1_C(D, c_d)$

$$\pi_{c_d} : \mathbf{\Omega}(D, c_d) \longrightarrow C$$

Darboux coordinates

- for (E, ∇) meromorphic connection, tr(∇) global section of $\Omega^1_C(D, c_d) \to C$
- affine isomorphism

$$\mathbf{\Omega}(D) \longrightarrow \mathbf{\Omega}(D, c_d); \quad (q, p) \longmapsto (q, p + \mathrm{tr}(\nabla)) = (q, \tilde{p})$$

- $\mathbf{\Omega}(D, c_d)$ is a symplectic surface with form $\mathrm{d}\tilde{p} \wedge \mathrm{d}q$
- accessory parameter of (E, ∇) at q_j

$$\tilde{p}_j = \mathrm{res}_{q_j}(\beta) + \mathrm{tr}(\nabla)|_{q_j},$$

- $\{(q_j, \tilde{p}_j)\}_{j=1}^N$ canonical coordinates of (E, ∇)

Coordinate map

- Let $M^0_{\mathrm{dR}} \subset M_{\mathrm{dR}}$ parameterize (E, ∇) such that $\dim_{\mathbb{C}} H^0(C, E) = 1$, B is reduced and $\mathrm{Supp}(B) \cap \mathrm{Supp}(D) = \varnothing$
- $\pi_{c_d,N} \colon \mathrm{Sym}^N(\mathbf{\Omega}(D, c_d)) \to \mathrm{Sym}^N(C)$ the map induced by the map $\pi_{c_d} \colon \mathbf{\Omega}(D, c_d) \to C$
- $$\Delta = \{q_{j_1} = q_{j_2} \text{ for some } j_1 \neq j_2\} \subset \mathrm{Sym}^N(C)$$
- $$\mathrm{Sym}^N(\mathbf{\Omega}(D, c_d))_0 := \pi^{-1}_{c_d,N}(\mathrm{Sym}^N(C \setminus \mathrm{Supp}(D)) \setminus \Delta).$$
- coordinate map
$$f_{\mathrm{App}} \colon M^0_X \to \mathrm{Sym}^N(\mathbf{\Omega}(D, c_d))_0$$
$$(E, \nabla) \mapsto \{(q_j, \tilde{p}_j)\}_{j=1}^N$$

Symplectic isomorphism

Atiyah–Bott, Bottacin–Markman, Boalch: M_{dR} is a holomorphic symplectic manifold of dimension $2N = 8g - 6 + 2n$.

Proposition

The map f_{App} is birational.

(Slight modification of independence and equality of dimensions.)
Symplectic form on $\mathrm{Sym}^N(\mathbf{\Omega}(D, c_d))$:

$$\omega = \sum_{j=1}^{N} \mathrm{d}\tilde{p}_j \wedge \mathrm{d}q_j$$

Theorem

The map f_{App} is symplectic.

Elliptic curve and divisors D, B

Fix $\lambda \in \mathbb{C} \setminus \{0, 1, \infty\}$,

- curve C obtained by gluing
  $U_0 := (y_1^2 - x_1(x_1 - 1)(x_1 - \lambda) = 0)$ with
  $U_\infty := (y_2^2 - x_2(1 - x_2)(1 - \lambda x_2) = 0)$, via identifying
  $x_1 = x_2^{-1}$ and $y_1 = y_2 x_2^{-2}$
- polar divisor $D = (t, s) + (t, -s)$ for fixed $t \in \mathbb{C}$
- case $t \notin \{0, 1, \lambda, \infty\}$: two logarithmic poles
- otherwise one irregular singularity of Poincaré–Katz rank 1
- $4 - 3 + 2 = 3$ points q_1, q_2, q_3 on C

$$q_j : (x_1, y_1) = (u_j, v_j)$$

such that $u_j \notin \{0, 1, \lambda, \infty, t\}$

Connection ∇_0

- $E_0 = \mathcal{O}_C \oplus (\Omega^1_C(D))^{-1}$
- Over U_0 with respect to a trivialization of $(\Omega^1_C(D))^{-1}$

$$\nabla_0 = \mathrm{d} + \begin{pmatrix} 0 & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$$

- where for some $\zeta_1, \zeta_2, \zeta_3, A_1, \ldots, B_3 \in \mathbb{C}$

$$\omega_{12} = \sum_{j=1}^{3} \frac{\zeta_j}{2} \cdot \frac{y_1 + v_j}{x_1 - u_j} \cdot \frac{\mathrm{d}x_1}{y_1} + \left(\frac{A_1 + A_2 y_1}{x_1 - t} + A_3 + A_4 x_1 \right) \frac{\mathrm{d}x_1}{y_1}$$

$$\omega_{21} := \frac{1}{x_1 - t} \frac{\mathrm{d}x_1}{y_1}$$

$$\omega_{22} := \sum_{j=1}^{3} \frac{1}{2} \cdot \frac{y_1 + v_j}{x_1 - u_j} \cdot \frac{\mathrm{d}x_1}{y_1} + \left(\frac{B_1 + B_2 y_1}{x_1 - t} + B_3 \right) \frac{\mathrm{d}x_1}{y_1}.$$

Fixing the polar parts – logarithmic case

- $t_1 = (t, s) \neq (r, -s) = t_2$
- fix complex numbers $\theta_1^{\pm}, \theta_2^{\pm}$ such that $\sum_{i=1}^{2}(\theta_i^{+} + \theta_i^{-}) = -1$
- impose eigenvalues of the matrix

$$\mathrm{res}_{t_1} \begin{pmatrix} 0 & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$$

  are given by $\theta_1^{+}, \theta_1^{-}$, and similarly for t_2

Lemma

There exist unique values of the parameters A_1, A_2, B_1, and B_2 such that the residues satisfy these constraints. Moreover, these parameter values are independent of u_1, u_2, u_3, ζ_1, ζ_2, and ζ_3.

Linear system – logarithmic case

The system to solve reads as

$$\frac{A_1 + A_2 s}{s} \cdot \frac{1}{s} = \theta_1^+ \cdot \theta_1^- \quad \text{and} \quad \frac{A_1 - A_2 s}{-s} \cdot \frac{1}{-s} = \theta_2^+ \cdot \theta_2^-$$

and

$$\frac{B_1 + B_2 s}{s} = \theta_1^+ + \theta_1^- \quad \text{and} \quad \frac{B_1 - B_2 s}{-s} = \theta_2^+ + \theta_2^-$$

This is clearly solvabe, and the solution is independent of u_1, u_2, u_3, ζ_1, ζ_2, and ζ_3.

Fixing the polar parts – irregular case

- for instance $t = 0$
- fix $\theta^{\pm}_{-2}, \theta^{+}_{-1} \in \mathbb{C}$ so that $\theta^{+}_{-2} \neq \theta^{-}_{-2}$
- set $\theta^{-}_{-1} = -1 - \theta^{+}_{-1}$ (Fuchs)

Lemma

There exist unique $A_1, A_2, B_1, B_2 \in \mathbb{C}$ such that the eigenvalues of

$$\mathrm{res} \begin{pmatrix} 0 & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$$

admit Laurent expansions of the form

$$\left(\theta^{\pm}_{-2} \frac{1}{y_1^2} + \theta^{\pm}_{-1} \frac{1}{y_1} + O(1) \right) \otimes \mathrm{d}y_1.$$

Moreover, these values are independent of u_i, ζ_i.

Linear system – irregular case

- locally C is given by $x_1 = h(y_1^2)$ for $h\colon U \to \mathbb{C}$, $h(0) = 0$
- $$\frac{\mathrm{d}x_1}{y_1} = \frac{2\,\mathrm{d}y_1}{3x_1^2 - 2(1+\lambda)x_1 + \lambda},$$
- $$\frac{\mathrm{d}x_1}{x_1 y_1} = \frac{\mathrm{d}y_1}{y_1^2} g(y_1^2) \qquad (g(0) = 2)$$
- these show

$$\omega_{12} = (A_1 + A_2 y_1)\frac{\mathrm{d}x_1}{x_1 y_1} + O(1) = 2(A_1 + A_2 y_1)\frac{\mathrm{d}y_1}{y_1^2} + O(1)$$

$$\omega_{21} = 2\frac{\mathrm{d}y_1}{y_1^2} + O(1)$$

$$\omega_{22} = (B_1 + B_2 y_1)\frac{\mathrm{d}x_1}{x_1 y_1} + O(1) = 2(B_1 + B_2 y_1)\frac{\mathrm{d}y_1}{y_1^2} + O(1).$$

Solution of linear system – irregular case

- We find

$$B_1 = \frac{1}{2}(\theta^+_{-2} + \theta^-_{-2}), \qquad B_2 = \frac{1}{2}(\theta^+_{-1} + \theta^-_{-1}) = -\frac{1}{2}.$$

- The quadratic equation

$$-\omega_{12}\omega_{21} = -4(A_1 + A_2 y_1)\frac{(\mathrm{d}y_1)^{\otimes 2}}{y_1^4} + O\left(\frac{1}{y_1^2}\right).$$

gives

$$A_1 = -\frac{1}{4}\theta^+_{-2}\theta^-_{-2}, \qquad A_3 = -\frac{1}{4}(\theta^+_{-2}\theta^-_{-1} + \theta^-_{-2}\theta^+_{-1}).$$

Apparent conditions

Lemma

The fact that ∇_0 *has apparent singular points at* q_1, q_2, q_3 *imposes 3 linear conditions on* A_3, A_4, B_3 *in terms of spectral data, and* $((u_j, v_j), \zeta_j)$*'s; we can uniquely determine* A_3, A_4, B_3 *from these conditions if, and only if, we have*

$$\det \begin{pmatrix} 1 & u_1 & \zeta_1 \\ 1 & u_2 & \zeta_2 \\ 1 & u_3 & \zeta_3 \end{pmatrix} \neq 0.$$

Vector bundle E

- $$\tilde{U}_0 := U_0 \setminus \{q_1, q_2, q_3\} \quad \text{and} \quad \tilde{U}_\infty := U_\infty \setminus \{q_1, q_2, q_3\}.$$
- tiny analytic open neighbourhoods $q_j \in \tilde{U}_{q_j}$
- $$B_{0q_j} := \begin{pmatrix} 1 & \frac{\zeta_j}{x_1 - u_j} \\ 0 & \frac{1}{x_1 - u_j} \end{pmatrix}$$
- $$B_{0\infty} := \begin{pmatrix} 1 & 0 \\ 0 & -x_2 \end{pmatrix}$$
- this cocycle $\rightsquigarrow$ E rank 2 holomorphic vector bundle

Connection ∇

- ∇_0 induces a connection ∇ on E
- ∇ has no singularity at q_j
- the canonical coordinates are q_j and $\tilde{p}_j = C\zeta_j + D$ for some $C, D \in \mathbb{C}$

Further questions

- extension over D and Δ
- generalization to higher rank
- application to isomonodromy