Total (gradient) variation regularization: exact support recovery and grid-free numerical methods

Romain Petit, joint work with Yohann De Castro and Vincent Duval

Off-the-grid methods for inverse problems in imaging, 22 November 2023

Inverse problems in imaging

Unknown image $u_0 : \mathbb{R}^2 \to \mathbb{R}$

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Obs. $y_0 = \Phi u_0 \in \mathcal{H}$

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Noisy obs. $y = y_0 + w$

Inverse problems in imaging

Unknown image $u_0 : \mathbb{R}^2 \to \mathbb{R}$

Obs. $y_0 = \Phi u_0 \in \mathcal{H}$

Inverse problem

Recover u_0 from y

Noisy obs. $y = y_0 + w$

Variational approach [Rudin et al., 1992, Chambolle and Lions, 1997]

The total (gradient) variation

$$\mathrm{TV}(u) \stackrel{\text{def.}}{=} \sup\left\{ -\int_{\mathbb{R}^2} u \,\mathrm{div}\phi \;\middle|\; \phi \in \mathrm{C}_c^\infty(\mathbb{R}^2, \mathbb{R}^2),\ \|\phi\|_\infty \le 1 \right\} \text{``}=\text{''} \int_{\mathbb{R}^2} |\nabla u|$$

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Solve

$$\min_{u \in \mathrm{L}^2(\mathbb{R}^2)} \frac{1}{2}\|\Phi u - y\|^2 + \lambda\, \mathrm{TV}(u) \qquad (\mathcal{P}_\lambda(y))$$

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noisy obs. y

solution (small λ)

solution (large λ)

Solve

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Variational approach [Rudin et al., 1992, Chambolle and Lions, 1997]

Representer th. [Boyer et al., 2019, Bredies and Carioni, 2019] + [Fleming, 1957]

Some sol. of $(\mathcal{P}_\lambda(y))$ are linear combinations of 1_E with *E simple*

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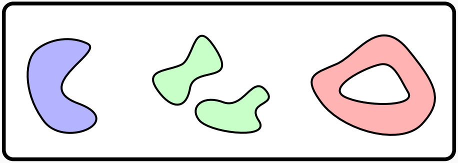

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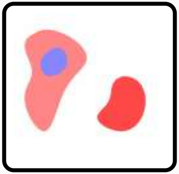
The sparse objects associated to TV are the piecewise constant functions



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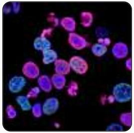
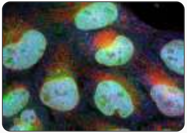
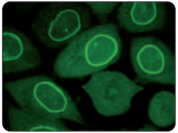
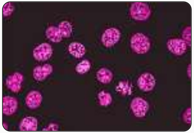

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unknown im. u_0

obs. $y_0 = \Phi u_0$

noisy obs. $y = y_0 + w$

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Noise robustness

Conv. of sol. to $(\mathcal{P}_\lambda(y_0 + w))$ when $\lambda, w \to 0$

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Numerical resolution of $(\mathcal{P}_\lambda(y))$

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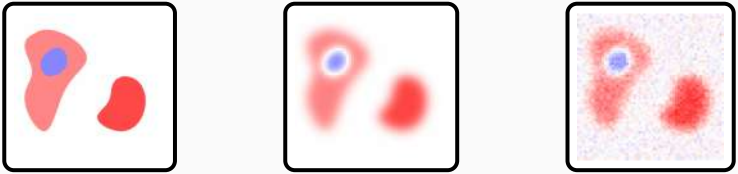

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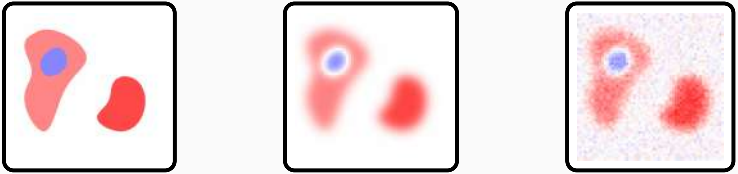

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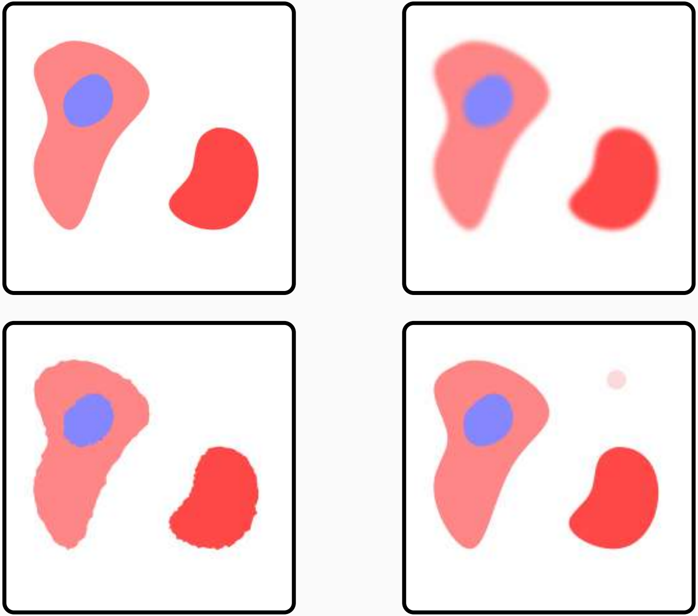
Assumptions

- u_0 piecewise constant
- $\mathrm{Im}(\Phi^*) \subset \mathrm{C}^1(\mathbb{R}^2)$

Noise robustness: exact support recovery

What kind of convergence?



First convergence result

$(\mathcal{P}_\lambda(\boldsymbol{y}_0 + \boldsymbol{w}))$

$$\min_{u \in \mathrm{L}^2(\mathbb{R}^2)} \mathrm{TV}(u) + \frac{1}{2\lambda} \|\Phi u - (y_0 + w)\|^2$$

$(\mathcal{P}_0(\boldsymbol{y}_0))$

$$\min_{u \in \mathrm{L}^2(\mathbb{R}^2)} \mathrm{TV}(u) \ \text{ s.t. } \ \Phi u = y_0$$

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Proposition [Chambolle et al., 2016, Iglesias et al., 2018]

If $\lambda_n \to 0$ and $\|w_n\| = \mathcal{O}(\lambda_n)$ (+ source cond.) then $u_n \to u_0$ (strictly in $\mathrm{BV}(\mathbb{R}^2)$) and

$$\left|U_n^{(t)} \Delta U_0^{(t)}\right| \to 0 \text{ and } \partial U_n^{(t)} \xrightarrow{\text{Hausdorff}} \partial U_0^{(t)} \text{ with } U^{(t)} = \begin{cases} \{u \geq t\} \text{ if } t \geq 0 \\ \{u \leq t\} \text{ otherwise.} \end{cases}$$

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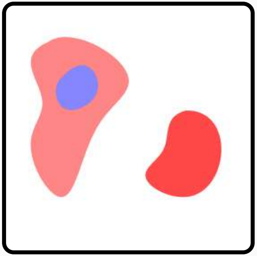
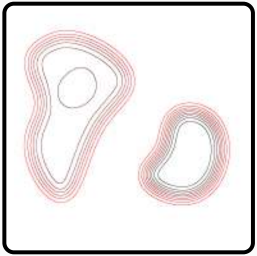


u_0

u_n



$U_n^{(t)}, \; t \in \mathbb{R}$

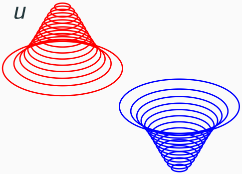
The prescribed curvature problem

Optimality condition (regularized pb.)

If u solves $(\mathcal{P}_\lambda(y_0 + w))$ then

- $\forall t > 0,\ \{u \geq t\}$ solves $(\mathcal{Q}(+\eta_{\lambda,w}))$
- $\forall t < 0,\ \{u \leq t\}$ solves $(\mathcal{Q}(-\eta_{\lambda,w}))$

for some $\eta_{\lambda,w}$



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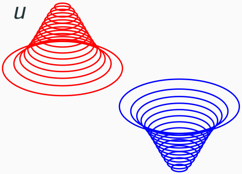

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Prescribed curvature problem

$$\min_{E\subset\mathbb{R}^2,\,|E|<+\infty} \operatorname{Per}(E) - \int_E \eta \qquad (\mathcal{Q}(\eta))$$

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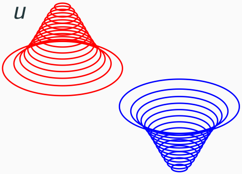

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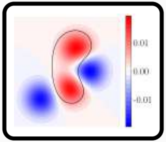

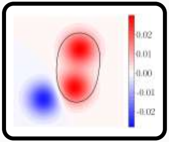

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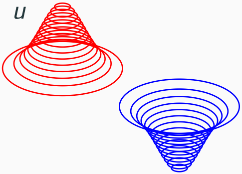

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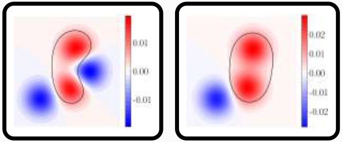

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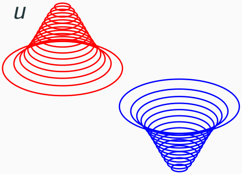

Convergence of curvature functionals

If $\lambda_n \to 0$ and $\frac{\|w_n\|}{\lambda_n} \to 0$ (+ source cond.) then $\eta_{\lambda_n,w_n} \to \eta_0$

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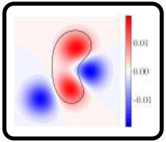

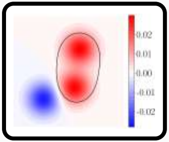

Optimality condition (constrained pb.)

We have that

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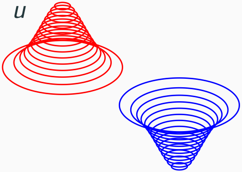

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Stability of the prescribed curvature problem

Assumptions

- η close to η_0 in $\mathrm{L}^2(\mathbb{R}^2)$ and $\mathrm{C}^1(\mathbb{R}^2)$
- $(\mathcal{Q}(\eta_0))$ has finitely many minimizers, all strictly stable

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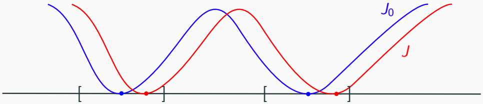
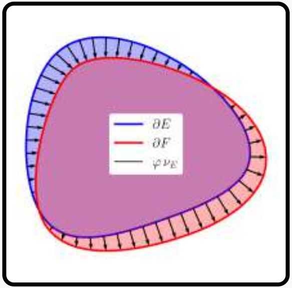
Proposition (informal)

Around each sol. E of $(\mathcal{Q}(\eta_0))$ there is exactly one sol. F of $(\mathcal{Q}(\eta))$ and

$$\partial F = (Id + \varphi\nu_E)(\partial E) \text{ with } \varphi \in \mathrm{C}^2(\partial E)$$





Exact support recovery

Assumptions

- $u_0 = \sum_{i=1}^{N} a_i 1_{E_i}$ with E_i simple and $\partial E_i \cap \partial E_j = \emptyset$ for every $i \neq j$
- non-degenerate source cond. $+$ injectivity cond.

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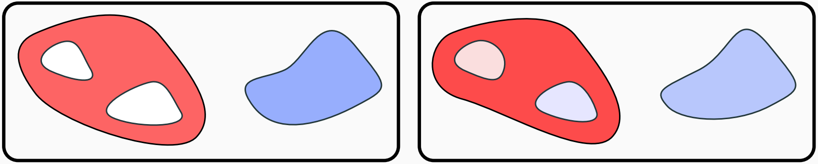

u_0 $\quad$ $u_{\lambda,w}$

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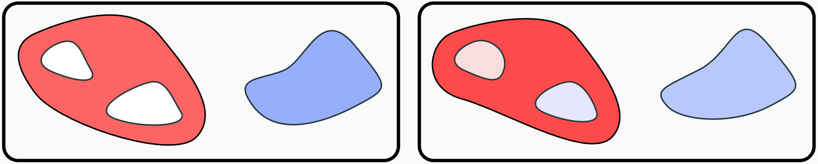


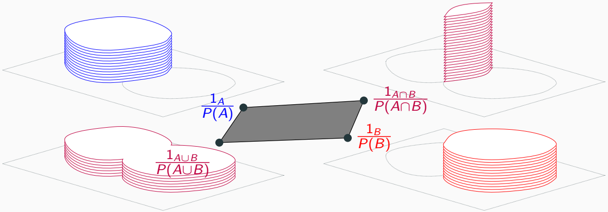
u_0 $u_{\lambda,w}$

Theorem

There exists $\alpha, \lambda_0 \in \mathbb{R}_+^*$ s.t., if $\lambda \leq \lambda_0$ and $\|w\|/\lambda \leq \alpha$, then

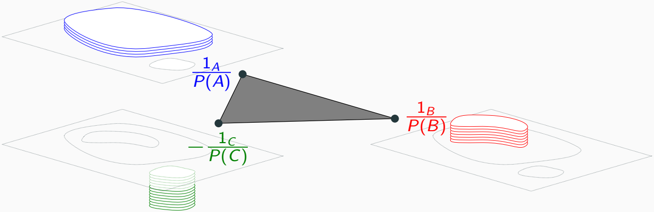
- $u_{\lambda,w} = \sum_{i=1}^N b_i 1_{F_i}$ with $\partial F_i = (Id + \varphi_i \nu_E)(\partial E)$
- $b_i \to a_i$ and $\|\varphi_i\|_{\mathrm{C}^2(\partial E_i)} \to 0$ when $\lambda, w \to 0$

The faces of the total (gradient) variation unit ball



Theorem

d-dimensional faces exposed by smooth enough functions are d-simplices



Numerical verif. of the non-degenerate source cond.

$\Phi u = h \star u$ with $h(x) = \exp(-\|x\|^2/(2\sigma^2))$

`github.com/rpetit/2023-support-recovery-tv`

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Deconvolution of a disk: $u_0 = a\,1_{B(0,R)}$

Condition satisfied if $\sigma \leq \sigma_0(R)$

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Deconvolution of radial images: $u_0 = a_1\,1_{B(0,R_1)} + a_2\,1_{B(0,R_2)}$

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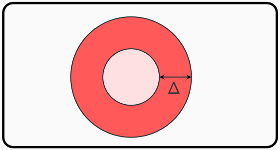
Deconvolution of radial images: $u_0 = a_1\, 1_{B(0,R_1)} + a_2\, 1_{B(0,R_2)}$

- If $\mathrm{signe}(a_1) = -\mathrm{signe}(a_2)$: need $|R_1 - R_2| > \Delta$



`github.com/rpetit/2023-support-recovery-tv`

Numerical verif. of the non-degenerate source cond.

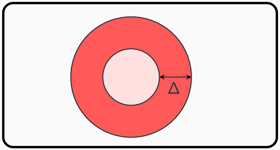
$\Phi u = h \star u$ with $h(x) = \exp(-\|x\|^2/(2\sigma^2))$

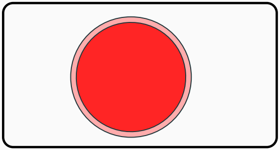
Deconvolution of a disk: $u_0 = a\,1_{B(0,R)}$

Condition satisfied if $\sigma \leq \sigma_0(R)$

Deconvolution of radial images: $u_0 = a_1\,1_{B(0,R_1)} + a_2\,1_{B(0,R_2)}$

- If $\mathrm{signe}(a_1) = -\mathrm{signe}(a_2)$: need $|R_1 - R_2| > \Delta$
- If $\mathrm{signe}(a_1) = \mathrm{signe}(a_2)$: super-resolution





github.com/rpetit/2023-support-recovery-tv

Numerical resolution: a grid-free approach

Numerical resolution of $(\mathcal{P}_\lambda(y))$

Solve

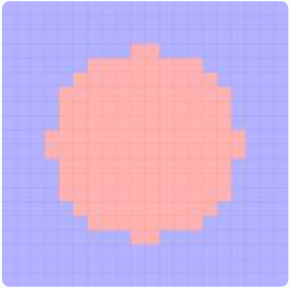
$$\min_{u \in \mathrm{L}^2(\mathbb{R}^2)} \frac{1}{2}\|\Phi u - y\|^2 + \lambda \, \mathrm{TV}(u) \qquad (\mathcal{P}_\lambda(y))$$

Solve

$$\min_{u \in \mathrm{L}^2(\mathbb{R}^2)} \frac{1}{2}\|\Phi u - y\|^2 + \lambda \, \mathrm{TV}(u) \qquad (\mathcal{P}_\lambda(y))$$

Fixed grid approximation

$$u = \sum_i \sum_j u_{ij} \, \mathbf{1}_{C_{ij}}$$



Discretizations of the total variation (images: [Tabti et al., 2018])

Anisotropic

- $\sum_{ij} |(D_x u)_{ij}| + |(D_y u)_{ij}|$
- Sharp edges, **grid bias**

Isotropic

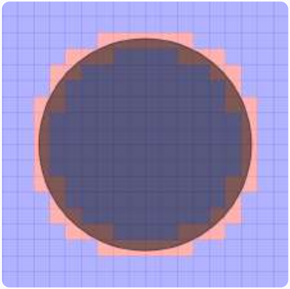
- $\sum_{ij} \sqrt{(D_x u)_{ij}^2 + (D_y u)_{ij}^2}$
- **Blur**

State of the art review: [Chambolle and Pock, 2021]

Numerical representation of simple images

Fixed grid

- $\mathcal{O}\left(1/h^2\right)$ pixels
- $\mathcal{O}\left(1/h\right)$ "relevant" pixels
- $u \mapsto \mathrm{TV}(u)$ convex



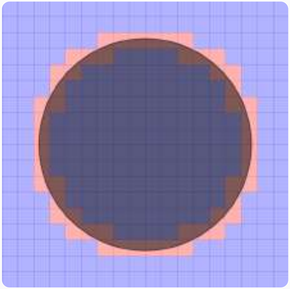
Numerical representation of simple images

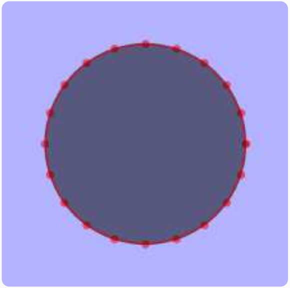
Fixed grid

- $\mathcal{O}\left(1/h^2\right)$ pixels
- $\mathcal{O}\left(1/h\right)$ "relevant" pixels
- $u \mapsto \mathrm{TV}(u)$ convex

Boundary discretization

- More efficient for simple img.
- Numerically more involved
- $E \mapsto \mathrm{TV}(1_E)$ "non convex"





Frank-Wolfe based algorithm

Algorithm

[Bredies and Pikkarainen, 2013]
[Boyd et al., 2017, Denoyelle et al., 2019]

Frank-Wolfe based algorithm

Algorithm

- $\eta_k = -\frac{1}{\lambda}\Phi^*(\Phi u_k - y)$

[Bredies and Pikkarainen, 2013]
[Boyd et al., 2017, Denoyelle et al., 2019]

Frank-Wolfe based algorithm

Algorithm

- $\eta_k = -\frac{1}{\lambda}\Phi^*(\Phi u_k - y)$
- $E_{k+1} \in \underset{E \text{ simple}}{\text{Argmax}} \; \frac{1}{P(E)} \left| \int_E \eta_k \right|$

[Bredies and Pikkarainen, 2013]
[Boyd et al., 2017, Denoyelle et al., 2019]

Frank-Wolfe based algorithm

Algorithm

- $\eta_k = -\frac{1}{\lambda}\Phi^*(\Phi u_k - y)$
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[Bredies and Pikkarainen, 2013]
[Boyd et al., 2017, Denoyelle et al., 2019]

Generalized Cheeger pb.

$$\underset{E\subset\mathbb{R}^2}{\text{Max.}} \quad \frac{1}{P(E)}\left|\int_E \eta\right|$$

$$\text{s.t.} \quad |E| < +\infty, \ 0 < P(E) < +\infty$$

Frank-Wolfe based algorithm

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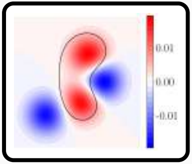

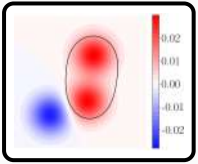

Frank-Wolfe based algorithm

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[Bredies and Pikkarainen, 2013]
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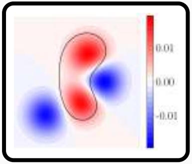

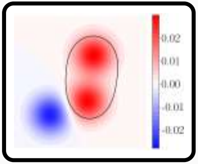

Frank-Wolfe based algorithm

Algorithm

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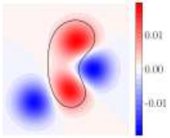
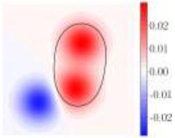
- $E_{k+1} \in \underset{E \text{ simple}}{\text{Argmax}} \frac{1}{P(E)} \left|\int_E \eta_k\right|$
- $u_{k+1} = \alpha_k \, u_k + \beta_k \, 1_{E_{k+1}}$

[Bredies and Pikkarainen, 2013]
[Boyd et al., 2017, Denoyelle et al., 2019]

Generalized Cheeger pb.

$$\underset{E\subset\mathbb{R}^2}{\text{Max.}} \quad \frac{1}{P(E)}\left|\int_E \eta\right|$$
$$\text{s.t.} \quad |E| < +\infty, \; 0 < P(E) < +\infty$$





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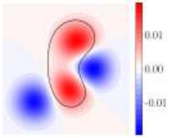
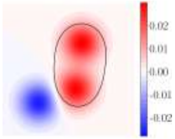




Iterates are linear combinations of indicator functions of simple sets

Frank-Wolfe based algorithm

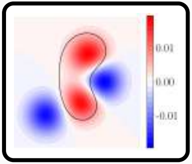
Algorithm

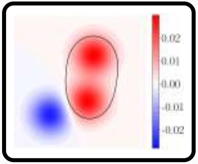
- $\eta_k = -\frac{1}{\lambda}\Phi^*(\Phi u_k - y)$
- $E_{k+1} \in \underset{E \text{ simple}}{\text{Argmax}} \frac{1}{P(E)} \left|\int_E \eta_k\right|$
- $u_{k+1} = \alpha_k \, u_k + \beta_k \, 1_{E_{k+1}}$
- Loc. opt. $(a, E) \mapsto F\left(\sum_i a_i 1_{E_i}\right)$

[Bredies and Pikkarainen, 2013]
[Boyd et al., 2017, Denoyelle et al., 2019]

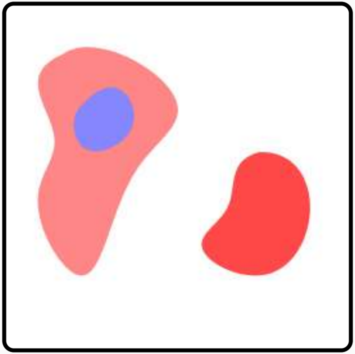
Generalized Cheeger pb.

$$\underset{E\subset\mathbb{R}^2}{\text{Max.}} \quad \frac{1}{P(E)}\left|\int_E \eta\right|$$
$$\text{s.t.} \quad |E| < +\infty, \; 0 < P(E) < +\infty$$

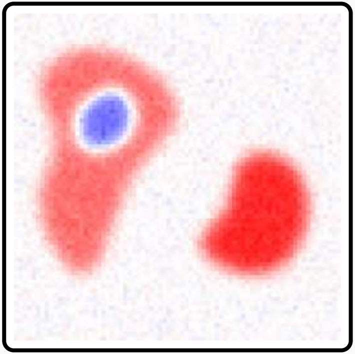




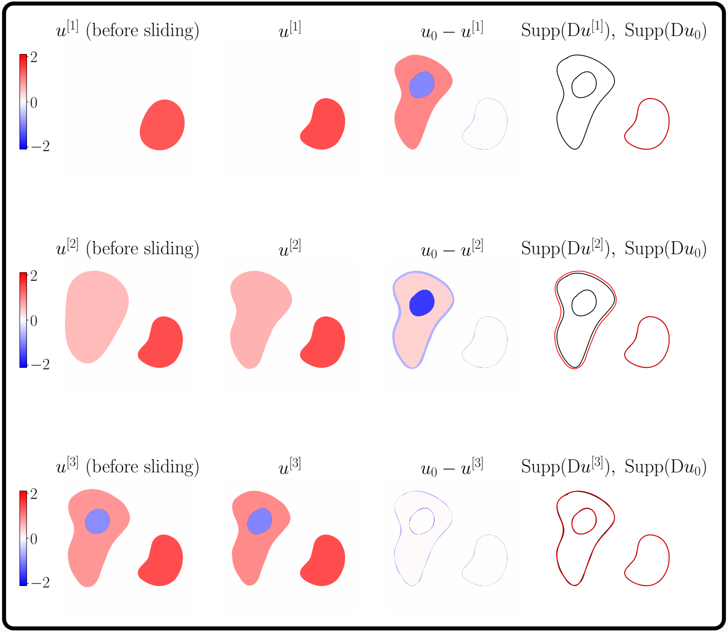
Iterates are linear combinations of indicator functions of simple sets

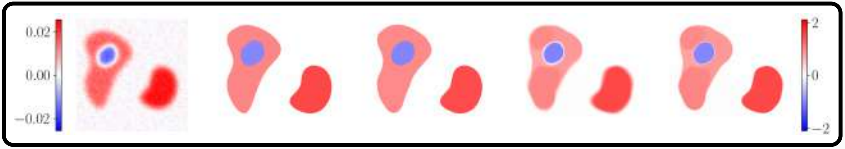


Unknown image u_0



Observations $y = \Phi u_0 + w$

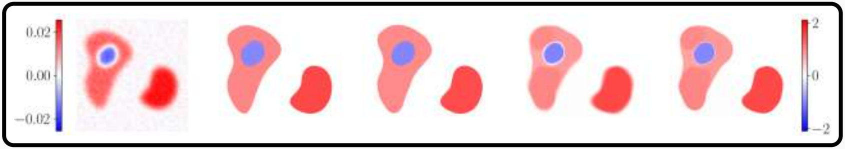

$u^{[1]}$ (before sliding)
$u^{[1]}$
$u_0 - u^{[1]}$
$\mathrm{Supp}(\mathrm{D}u^{[1]}),\ \mathrm{Supp}(\mathrm{D}u_0)$
$u^{[2]}$ (before sliding)
$u^{[2]}$
$u_0 - u^{[2]}$
$\mathrm{Supp}(\mathrm{D}u^{[2]}),\ \mathrm{Supp}(\mathrm{D}u_0)$
$u^{[3]}$ (before sliding)
$u^{[3]}$
$u_0 - u^{[3]}$
$\mathrm{Supp}(\mathrm{D}u^{[3]}),\ \mathrm{Supp}(\mathrm{D}u_0)$
2
0
−2



Left to right: observations, signal, ours, isotropic TV, “Condat’s” TV

Numerical results

github.com/rpetit/tvsfw

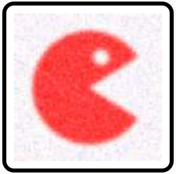


Left to right: observations, signal, ours, isotropic TV, "Condat's" TV

Signal u_0

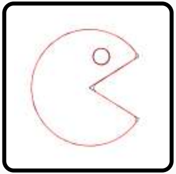
Isotropic TV

Ours $\hat{u}_{\lambda,w}$



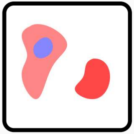
Observations

"Condat's" TV

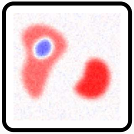


$\mathrm{Supp}(\mathrm{D}u_0)$, $\mathrm{Supp}(\mathrm{D}\hat{u}_{\lambda,w})$

Perspectives (i)



u_0



$\Phi u_0 + w$

$\hat{u}_{\lambda,w}$

$\hat{u}_{\lambda,w} - u_0$

$\mathrm{D}u_0$, $\mathrm{D}\hat{u}_{\lambda,w}$

Perspectives (i)

u_0 $\Phi u_0 + w$ $\hat{u}_{\lambda,w}$ $\hat{u}_{\lambda,w} - u_0$ $\mathrm{D}u_0$, $\mathrm{D}\hat{u}_{\lambda,w}$

Recovery guarantees

- Non-degenerate source cond.
- Implicit bias of TV reg.

Perspectives (i)

u_0 $\Phi u_0 + w$ $\hat{u}_{\lambda,w}$ $\hat{u}_{\lambda,w} - u_0$ $\mathrm{D}u_0$, $\mathrm{D}\hat{u}_{\lambda,w}$

Recovery guarantees

- Non-degenerate source cond.
- Implicit bias of TV reg.

Numerical resolution

- Robust sliding step (topology changes)

Perspectives (i)

u_0 $\Phi u_0 + w$ $\hat{u}_{\lambda,w}$ $\hat{u}_{\lambda,w} - u_0$ $\mathrm{D}u_0$, $\mathrm{D}\hat{u}_{\lambda,w}$

Recovery guarantees

- Non-degenerate source cond.
- Implicit bias of TV reg.

Numerical resolution

- Robust sliding step (topology changes)
- "Continuous" TV reg. benchmark

Benchopt

Perspectives (i)

u_0 | $\Phi u_0 + w$ | $\hat{u}_{\lambda,w}$ | $\hat{u}_{\lambda,w} - u_0$ | $\mathrm{D}u_0$, $\mathrm{D}\hat{u}_{\lambda,w}$

Recovery guarantees

- Non-degenerate source cond.
- Implicit bias of TV reg.

Numerical resolution

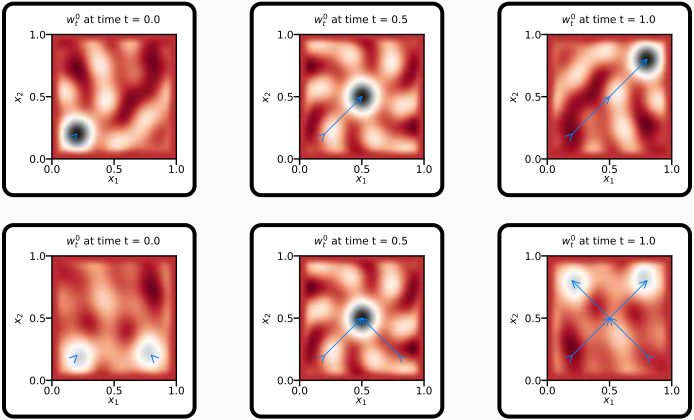
- Robust sliding step (topology changes)
- “Continuous” TV reg. benchmark
- Applications (cell im., piecewise homog. textures)

Benchopt

Perspectives (ii)

OT reg. for dynamic inverse problems [Bredies et al., 2022]

- Recover $t \mapsto \sum_i a_i(t)\delta_{x_i(t)}$
- Support stability, implicit bias, fast reconstruction?



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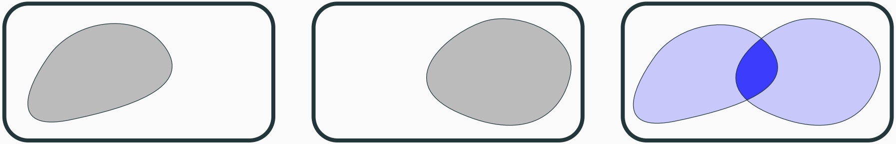
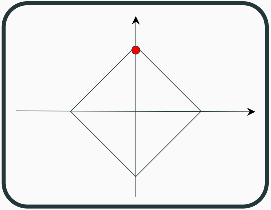
Piecewise constant functions

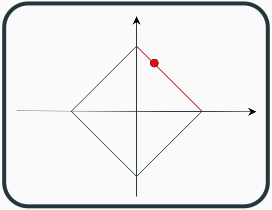


E $\qquad$ F $\qquad$ $u = 1_E + 1_F$

Sparsity with respect to R

f is s-sparse if smallest face of $\{R \leq R(f)\}$ containing f has dimension $s - 1$





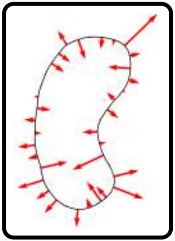
Two-step approximation of generalized Cheeger sets

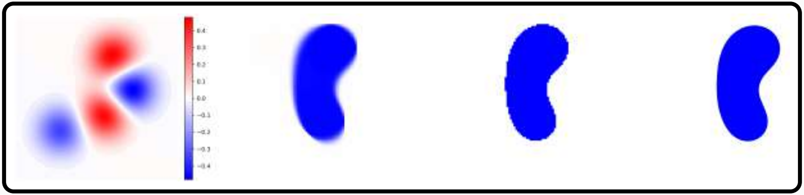
Fixed grid initialization [Carlier et al., 2009]

Solve $\min_{u \in E^h} \langle \eta^h, u \rangle$ s.t. $\mathrm{TV}^h(u) \leq 1$

Shape gradient algorithm

- $\theta_n \in \underset{\theta \in \Theta}{\mathrm{Argmax}} \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left[J\left((Id + \epsilon\, \theta)(E_n)\right) - J(E_n) \right]$
- $E_{n+1} = (Id + \epsilon_n\, \theta_n)(E_n)$





Implementation: `github.com/rpetit/PyCheeger`

Topology changes during the local descent