New higher-spin topological systems in 3D

Strange higher-spins are wild quivers ?

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[to appear soon]

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1. Introduction
2. Higher dualisations of linearised gravity and Maxwell's
3. First-order reformulation of the spin 2-spin 3 systems
4. Generalisations

1 Introduction

- Fierz-Pauli programme → all possible off-shell descriptions

  of spin-s massless field ?

  ↳ Interactions may not choose the most economical description

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Higher-spin Gravity

Hidden symmetries of Gravity

Non-linear realisation of $e_{11} \ltimes \ell_1$

Higher, or "exotic" descriptions

• Electric-magnetic duality, perhaps as fundamental as Lorentz symmetry.

In non-Abelian theory, relates strong and weak coupling regimes. [Long story: Heavyside, Dirac, ...]

• For spin-2 (linearized), studied by P. West, Hull (2001). Previous attempts in the massive case by Curtright & Freund in 80's. Further studied in 2002, on-shell, by X. Bekaert & N.B.

• First, review for spin (better, helicity) one.

- On-shell duality in electromagnetism in vacuum

Maxwell's equations $\boxed{\begin{array}{l}\partial^\mu F_{\mu\nu} = 0 \\ \partial_{[\mu} F_{\nu\rho]} = 0\end{array}}$ $(-\mu_0 J_\nu$ when sources$)$ (Field equations)

(Bianchi identities) $\quad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu = 2\,\partial_{[\mu} A_{\nu]}$,

Rewritten as $\boxed{*d*F_{[2]} = 0 \qquad \& \qquad dF_{[2]} = 0}$

where
- $F_{[2]} = \frac{1}{2} dx^\mu \wedge dx^\nu \; F_{\mu\nu}$ Faraday 2-form , $\quad dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$
  $= dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$
  $\in T^*M \otimes T^*M$.
- $d \; : \; \Omega^p(M) \to \Omega^{p+1}(M)$ exterior derivative , $d^2 = 0$
- $*d* \; : \; \Omega^{p+1}(M) \to \Omega^p(M)$ co-differential

Hodge dual : $*(dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}) = \frac{1}{(n-p)!} \epsilon^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_{n-p}} \, dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{n-p}}$, $\quad *^2\big|_{\Omega^p} = (-)^{p(n-p)+1} \, Id\big|_{\Omega^p}$

where $\epsilon^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_{n-p}} = \eta_{\nu_1 \rho_1} \cdots \eta_{\nu_{n-p} \rho_{n-p}} \epsilon^{\mu_1 \dots \mu_p \rho_1 \dots \rho_{n-p}}$ (in Minkowski space M)

Duality : $\boxed{F_{[2]} \mapsto *F_{[2]}}$ i.e. $\begin{pmatrix} \vec{E} \\ c\vec{B} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \vec{E} \\ c\vec{B} \end{pmatrix}$

Bianchi identity $\leftrightarrow$ Field equations

• On-shell duality for massless spin-2 field

Use condensed notation (X. Bekaert & N.B., 2002)

EI : $\mathrm{Tr}\, K \approx 0 \iff K^{\mu}{}_{\alpha\mu\nu} \approx 0$, where $K = d^{(1)} d^{(2)} h$,

BI : $\mathrm{Tr}_{12} *_1 K \equiv 0 \iff K_{[\mu\nu|\varrho]\sigma} \equiv 0$,

EII : $d^{\dagger} K \approx 0 \iff \partial^{\mu} K_{\mu\nu\varrho\sigma} \approx 0$,

BII : $d K \equiv 0 \iff \partial_{[\mu} K_{\nu\varrho]|\alpha\beta} \equiv 0$.

• $K \equiv K_{[2,2]} = \frac{1}{4}\, d^{(1)}x^{\mu}\, d^{(1)}x^{\nu}\, d^{(2)}x^{\alpha}\, d^{(2)}x^{\beta}\, K_{\mu\nu|\alpha\beta}$

• dual $d^{\dagger}_{(i)} x^{\mu}$ s.t. $\boxed{\{ d^{(i)}x^{\mu} , d^{\dagger}_{(i)} x^{\nu} \} = \eta^{\mu\nu}}$,

• $d^{(i)} := d^{(i)}x^{\mu} \frac{\partial}{\partial x^{\mu}}$, $d^{\dagger}_{(i)} := d^{\dagger}_{(i)} x^{\mu} \frac{\partial}{\partial x^{\mu}}$,

• $\mathrm{Tr}_{ij} = \eta_{\mu\nu}\, d^{\dagger}_{(i)} x^{\mu}\, d^{\dagger}_{(j)} x^{\nu}$

As operators : $\qquad | K_{[2,2]} \rangle = \frac{1}{4}\, d^{(1)}x^{\mu}\, d^{(1)}x^{\nu}\, d^{(2)}x^{\alpha}\, d^{(2)}x^{\beta}\, K_{\mu\nu|\alpha\beta}\, |0\rangle$

$$d^{\dagger}_{(i)} x^{\mu}\, |0\rangle \overset{!}{=} 0 \qquad \text{destruction} .$$

$$\left[d^{(i)}x^{\mu} , d^{(j)}x^{\nu} \right]_{\mathbb{Z}_2} = 0 \quad , \quad \left[d^{(i)}x^{\mu} , d^{\dagger}_{(j)}x^{\nu} \right]_{\mathbb{Z}_2} = \delta^{i}_{j}\, \eta^{\mu\nu} .$$

<u>Twisted-duality relations</u> $\qquad K \longmapsto *_1 K \quad , \quad *_1 K \longmapsto -K$

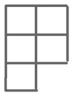
$$\vec{K} := \begin{pmatrix} K \\ *_1 K \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} K \\ *_1 K \end{pmatrix} = J\vec{K} \quad , \quad J = \pi/2 \ \text{rotation} .$$

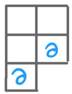
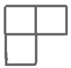
$$\rightsquigarrow \begin{pmatrix} E\,\mathrm{I} \\ E\,\mathrm{II} \end{pmatrix} \longleftrightarrow \begin{pmatrix} B\,\mathrm{I} \\ B\,\mathrm{II} \end{pmatrix} \quad \text{under duality} .$$

<u>Example, $n = 5$</u> :

$$\left.\begin{array}{ll} \underline{B\mathrm{I}} : & \mathrm{Tr}_{12} *_1 K \equiv 0 \\ \underline{B\mathrm{II}} : & d\, K \equiv 0 \end{array}\right\} \Longrightarrow K_{[2,2]} = d^{(1)} d^{(2)} h_{[1,1]} \quad , \quad h \sim$$ 

$\tilde{K} := *K$

$(\underline{E\mathrm{I}})$: $\mathrm{Tr}\, K = 0 \underset{\text{duality}}{\Longleftrightarrow} \mathrm{Tr} *_1 \tilde{K} = 0 \ (\tilde{B\mathrm{I}}) \Longleftrightarrow \tilde{K} \sim$ 

$(\underline{E\mathrm{II}})$: $d^{\dagger} K = 0 \Longleftrightarrow d\,\tilde{K} = 0 \ (\tilde{B\mathrm{II}}) \Longleftrightarrow \tilde{K} \sim$  $, \ C \sim$  $\equiv \tilde{h}$

② (Higher) dualisations of linearised gravity and Maxwell's : OFF_SHELL

An off-shell dualisation was initiated in 2001 by P. West, completed in 2003 by N.B., S. Cnockaert and M. Henneaux.

In [N.B., P. Cook, D. Ponomarev_2012] $\Longrightarrow$ Other off-shell dualisations schemes proposed

First, review the dualisation of [West, N.B. - Cnockaert - Henneaux]

Parent action $\quad S[Y^{abc|}{}_d, \omega_{ab|c}] = \int d^n x \left(\text{"}\omega\,\omega\text{"} + \partial_a \omega_{bc|}{}^d \, Y^{abc|}{}_d \right)$

$$Z = \int \mathcal{D}\omega \, \underline{\underline{\mathcal{D}Y}} \, \exp \tfrac{i}{\hbar} S[\omega, Y]$$

enforces $\quad \omega_{ab|c} = \partial_{[a} e_{b]c}$

semi_classical $\quad S[e_{ab}] = $ Fierz_Pauli

with local Lorentz $\quad \delta e_{ab} = \lambda_{ab}$

Field $\omega_{ab|c}$ auxiliary

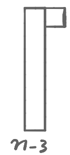
$$\frac{\delta S}{\delta \omega} \approx 0 \quad \Longrightarrow \quad \omega_{ab|c} \sim \partial^d Y_{abd|c}$$

$$S[Y^{abc|}{}_d] = \int d^n x \left(\partial^\circ Y_{ab\circ|c} \, \partial_\bullet Y^{\bullet ab|c} + \dots \right)$$

$$Y^{abc|}{}_e = \frac{1}{(n-3)!} \, \varepsilon^{abc\, d_1 \dots d_{n-3}} \, C_{d[n-3]|e}$$

Gauge inv. $\quad \delta_\lambda Y_{abc|}{}^e = \delta^e_{[a} \lambda_{bc]} \quad \Longrightarrow \quad \delta_\lambda C_{a[n-3]|b} = \varepsilon_{a[n-3]bcd} \, \lambda^{cd}$

$$C_{[n-3,1]} \rightsquigarrow C_{a_1 \dots a_{n-3}, b} = C_{[a_1 \dots a_{n-3}], b} \quad \text{s.t.} \quad C_{[a_1 \dots a_{n-3}, b]} \equiv 0$$

i.e. $C_{[n-3,1]} \sim$ 

of $GL(n)$ appears in Minkowski spacetime $\mathbb{R}^{1,n-1}$

that propagates the d.o.f. of Fierz-Pauli's graviton h_{ab} with $\eta^{bd} K_{ab,cd}(h) = 0$.

. We discussed: Hull's [2001] twisted on-shell duality

relating
$$K_{a_1 \dots a_{n-2},}{}^{b_1 b_2}(C) := \partial_{[a_1} \partial^{[b_1} C_{a_2 \dots a_{n-2}],}{}^{b_2]}$$

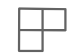
to
$$K_{ab,}{}^{cd}(h) := -\tfrac{1}{2} \, \partial_{[a} \partial^{c} h^{d]}{}_{b]}$$

via
$$\boxed{K_{[n-2,2]}(C) = *_1 K_{[2,2]}(h)}$$

2.1) Higher dual of vector field in dimensions 4 & 3

Idea : A_b viewed as a $A_{[0,1]}$ bi-form

$$A_{[0,1]} \xrightarrow[\text{higher dualise}]{} C_{[n-0-2,1]} \underset{n=4}{=} C_{[2,1]} \sim$$



$$\underset{n=3}{=} h_{[1,1]} \sim$$

- Starts from Maxwell and IBP : $S[A_a] = -\frac{1}{2}\int d^n x \left(\partial_a A_b \, \partial^a A^b - \partial_a A^a \, \partial_b A^b\right)$
- Parent action $S[Y^{ab|}{}_c, P_{a|}{}^b] = \int d^n x \left(P_{a|b} \, \partial_c Y^{ca|b} - \frac{1}{2} P_{a|b} P^{a|b} + \frac{1}{2} P^{a|}{}_a \, P_{b|}{}^b \right)$

$$\cdot \quad \frac{\delta S[Y,P]}{\delta P_{a|b}} \approx 0 \iff P^{a|b} \approx \partial_c Y^{ca|b} - \eta^{ab} \frac{1}{n-1} \partial_c Y^{cd|}{}_d$$

substitute to get

$$\boxed{S[Y^{ab|}{}_c] = \int d^n x \left[\frac{1}{2} \partial_c Y^{ca|b} \, \partial_d Y^d{}_{a|b} - \frac{1}{2(n-1)} \partial_a Y^{ab|}{}_b \right]}$$

. From $$\boxed{S[Y^{ab|}{}_c] = \int d^n x \left[\tfrac{1}{2}\, \partial_c Y^{ca|b}\, \partial_d Y^d{}_{a|b} - \frac{1}{2(n-1)}\, \partial_a Y^{ab|}{}_b \right]}$$

invariant under $$\delta Y^{ab|}{}_c = \delta_c^{[a}\, \partial^{b]} \lambda + \partial_d \tau^{abd|}{}_c \quad ,$$

one decomposes $$Y^{ab|}{}_c = X^{ab|}{}_c + \delta_c^{[a}\, Z^{b]} \quad , \qquad X^{ab|}{}_b \equiv 0$$

irreducibly under $Gl(n)$.

Ⓐ <u>Hodge-dualise in 4D</u> : $X^{ab|}{}_c \overset{*_1}{\longleftrightarrow} T_{ab|c} \sim \begin{ytableau} a & c \\ b \end{ytableau}$ that gauge transforms as

with $$\left\{ \begin{array}{l} \delta\, \ydiagram{2,1} = {}^{S}\begin{ytableau} \, & \, \\ \partial \end{ytableau} + {}^{A}\begin{ytableau} \, & \partial \\ \, \end{ytableau} \\[2ex] \delta Z_a = \partial_a \lambda + \partial^b A_{ab} \end{array} \right.$$ for the vector.

Perform some change of field variables and dualize $Z_a \leftrightarrow \tilde{A}_a$ to get

$$\boxed{S[T_{ab|c}, \tilde{A}_a] = \int d^4x \left[\mathcal{L}^{\text{Curt.}}(T_{ab,c}) + \frac{1}{4} F^{ab}(\tilde{A}) F_{ab}(\tilde{A}) + \frac{1}{2} \tilde{A}^a K''_a(T) \right]}$$

where $K^{a[3]}{}_{b[2]} := 6\, \partial^{[a} \partial_{[b} T^{aa],}{}_{b]}$ curvature , $K'' := \mathrm{Tr}^2 K$.

The gauge invariances are the ones expected for a Curtright field and a vector.

The field equations give
$$-\partial_a F^{ab}(\tilde{A}) + \frac{1}{2} K''^{b} = 0 \qquad (1)$$

$$K'^{ab]}{}_c + \delta_c^{[a} K''^{b]} - \frac{1}{2} \partial_c F^{ab} - \delta_c^{[a} \partial_d F^{b]d} = 0 \qquad (2)$$

Take the trace of (2), combine with (1) to get the equations of motion and duality relation $\boxed{d^{\dagger} F_{[2]} = 0 = \mathrm{Tr}^2 K_{[3,2]} \quad \& \quad \mathrm{Tr}\, K_{[3,2]} = d_2 F_{[2,0]}}$ $\Rightarrow$ no doubling of d.o.f. !

(B) Hodge-dualise in 3D

$S[Y^{ab|}{}_c]$ with $Y^{ab|}{}_c = \varepsilon^{abd} h_{cd} + 2\,\delta_c^{[a} Z^{b]}$, $h_{ab} = h_{ba}$, giving an action $S[h_{ab}, Z_a]$

$$S[h_{ab}, Z_a] = \int d^3x \left[-\tfrac{1}{2}\,\partial_a h_{bc}\,\partial^a h^{bc} + \tfrac{1}{2}\,\partial_a h_{bc}\,\partial^b h^{ac} + \tfrac{1}{2}\,\varepsilon^{bcd}\,\partial^a h_{ab}\,F_{cd}(Z) + \tfrac{1}{4}\,F^{ab}(Z)\,F_{ab}(Z) \right]$$

invariant under $\delta h_{ab} = 2\,\partial_{(a}\epsilon_{b)}$, $\delta Z_a = \partial_a \lambda + \varepsilon_{abc}\,\partial^b \epsilon^c$

mixing

- Dualise the vector Z_a in 3D to a scalar. After a field redefinition, one finds

$$S[h_{ab}, \phi] = \int d^3x \Big[-\tfrac{1}{2}\,\partial_a h_{bc}\,\partial^a h^{bc} + \tfrac{1}{2}\,\partial_a h\,\partial^a h - \partial_a h\,\partial_b h^{ab} + \partial_a h^{ab}\,\partial^c h_{bc}$$
$$+ \tfrac{1}{2}\,\partial_a \phi\,\partial^a \phi + \partial_a \phi\,(\partial_b h^{ab} - \partial^a h) \Big] \, .$$

As consequence of field equations : $\boxed{\Box\phi \approx 0 \approx R(h) \quad \& \quad R_{ab}(h) \approx \partial_a \partial_b \phi}$

Field equations for propagation & duality relation = no doubling

2.2) Higher dualisations of Maxwell's in 3D

- Maxwell in 3D $\sim$ massless scalar in 3D : 1 propagating d.o.f.

- Start from the dual action obtained by the second higher dualisation of scalar theory,

  i.e. the first higher dualisation of Maxwell's vector reviewed above

$$S[h_{ab}, Z_a] = \int d^3x \left[-\tfrac{1}{2}\partial_a h_{bc}\,\partial^a h^{bc} + \tfrac{1}{2}\partial_a h_{bc}\,\partial^b h^{ac} + \tfrac{1}{2}\varepsilon^{bcd}\,\partial^a h_{ab}\,F_{cd}(Z) + \tfrac{1}{4}F^{ab}(Z)\,F_{ab}(Z)\right]$$

We perform the higher dualisation $\mathcal{D}$ of h_{ab} via parent action $S[G_{a|bc}, D_{ab|}{}^{cd}, A_a]$,

eliminate the auxiliary field $G_{a|bc}$ to obtain $S[D_{ab|}{}^{cd}, A_a]$. As before, $\tilde{D}^{a|ij} := -\tfrac{1}{2}\varepsilon^{abc} D_{bc|}{}^{ij}$.

Perform a field redefinition $\{\underset{\underline{18}}{\tilde{D}_{a|cd}}, \underset{\underline{3}}{A_a}\} \longleftrightarrow \{\underset{\underline{10}}{\phi_{abc}}, \underset{\underline{5}}{f_{ab}}, \underset{\underline{3}}{U_a}, \underset{\underline{3}}{A_a}\}$ where $f_{ab} \sim \Box\!\Box$ of $SO(3)$.

s.t. U_a enters the action only via $F_{ab}(U) = 2\,\partial_{[a}U_{b]}$. Dualise U_a in 3D to a scalar σ that one adds

to $f_{ab} \longrightarrow h_{ab}$ traceful.

The final action is invariant under

$$\begin{cases} \delta \phi_{abc} = 3\, \partial_{(a} \xi_{bc)} \\[1em] \delta h_{ab} = 2\, \partial_{(a} \epsilon_{b)} + 2\, \varepsilon_{pq(a}\, \partial^p \xi^q{}_{b)} \\[1em] \delta A_a = \frac{3}{2}\, \partial_a \xi + \varepsilon_{abc}\, \partial^b \epsilon^c \end{cases}$$

<u>Rem</u> : $\delta\big(\underbrace{\phi_{abc} - \frac{2}{3}\eta_{(ab}A_{c)}}_{\varphi_{abc}}\big) = 3\, \partial_{(a} \hat{\xi}_{bc)} - \frac{2}{3}\eta_{(ab}\, \varepsilon_{c)pq}\, \partial^p \epsilon^q$.

<u>Field equations</u> : seemingly too many propagating d.o.f. since one finds that

- $\bar{\bar{K}}_{ab}(\phi) \approx 0$ where $\bar{\bar{K}}_{ab} := \eta^{cd}\eta^{ij} K_{ab|ci|dj}$ and $K_{ab|cd|ef} := 8 \begin{array}{|c|c|c|} \hline a & c & e \\ \hline \partial_b & \partial_d & \partial_f \\ \hline \end{array}$ curvature of ϕ_{abc} .

- $\tilde{R}^{ab|}{}_{ab} \approx 0$ where $\tilde{R}_{ab|}{}^{cd} = K_{ab|}{}^{cd}(h) - 2\left(\varepsilon_{abm}\partial^{[c}\Psi^{d]m} + \varepsilon^{cdm}\partial_{[a}\Psi_{b]m}\right)$, $\Psi^a{}_b := \partial_b \phi^a - \partial^c \phi^a{}_{bc}$

  $\hookrightarrow$ Riemann-like curvature for h_{ab} .

- $\partial_a \tilde{F}^{ab} \approx 0$ where $\tilde{F}_{ab} := F_{ab}(A) + \partial_{[a}\phi_{b]c}{}^c + \varepsilon_{abc}\left(\partial_d h^{cd} - \partial^c h\right)$ fieldstrength for A_a

However, combining the field equation, one finds the duality relations

$$\boxed{\begin{aligned} K_{ab|cd|ef}(\phi) &\approx -\frac{8}{21}\,\varepsilon_{efq}\,\partial^q\,\tilde{R}_{ab|cd} \\[2em] \tilde{R}^{ab|}{}_{cd} &\approx \frac{7}{4}\,\varepsilon_{cdm}\,\partial^m\,\tilde{F}^{ab} \end{aligned}}$$

$$\Longrightarrow \quad K_{ab|cd|ef}(\phi) \approx -\frac{2}{3}\,\varepsilon_{cdp}\,\varepsilon_{efq}\,\partial^p\partial^q\,\tilde{F}_{ab} \quad ,$$

All the duality relations and E.o.M come out of the action.

2.3) Higher dualisation of the graviton in 3D

Gravity in 3D is topological. Dual formulation thereof by higher dualisation → higher spin

- $$S[G_{a|bc}, D_{ab|}{}^{cd}] = \int d^3x \left[-\tfrac{1}{2} G_{a|bc} G^{a|bc} + \tfrac{1}{2} G_{a|c}{}^{c} G^{a|b}{}_{b} - G_{a|}{}^{ab} G_{b|c}{}^{c} + G_{a|}{}^{ab} G^{c|}{}_{cb} + G^{d}{}_{|bc} \partial^a D_{ad|}{}^{bc} \right]$$

$$G_{a|bc} = G_{a|cb} \,, \quad D_{ab|}{}^{cd} = -D_{ba|}{}^{cd} = -D_{ba|}{}^{dc}$$

Gauge-invariant under

$$\delta G^{a}{}_{|bc} = 2\, \partial^a \partial_{(b} \epsilon_{c)} \,, \quad \delta D_{ab|}{}^{cd} = \varepsilon_{abp} \partial^p v^{cd} + 2\, \eta^{cd} \partial_{[a} \epsilon_{b]} + 4\, \delta^{(c}_{[a} \partial_{b]} \epsilon^{d)} .$$

- $\dfrac{\delta S}{\delta D} \approx 0 \implies G_{a|bc} \approx \partial_a h_{bc}$ with $h_{ab} = h_{ba}$ symmetric

↳ $$S[\partial_a h_{bc}, D_{ab|}{}^{cd}] = \int d^3x \left[-\tfrac{1}{2} \partial_a h_{bc} \partial^a h^{bc} + \tfrac{1}{2} \partial_a h \partial^a h - \partial_a h^{ab} \partial_b h + \partial_a h^{ab} \partial^c h_{bc} \right]$$

Fierz-Pauli

- $\dfrac{\delta S}{\delta G_{a|bc}} \approx 0 \implies G_{a|bc} \approx \partial^d D_{da|bc} + \partial D =: G_{a|bc}(\partial D)$

↳ $S[G_{a|bc}(\partial D), D_{ab|}{}^{cd}] =: S[D_{ab|}{}^{cd}]$ Dual action

$$D_{ab|}{}^{pq} =: \varepsilon_{abm}\, \tilde{D}^{m|pq} \quad , \quad \tilde{D}^{a|cd} = \tilde{\varphi}^{acd} + 2\, \varepsilon^{ab(c}\, Z_{b|}{}^{d)} \quad , \quad Z_{a|}{}^{a} \equiv 0$$

$Gl(3)$ decomposition

- Further decompose under $SO(3)$:

$$\tilde{D}^{a|bc} \sim \Box \otimes \left(\Box\Box \oplus \bullet \right) \simeq \Box\Box\Box \oplus \Box\Box \oplus \Box \oplus \Box \qquad \in SO(3)$$

- A linear combination of the two vectors can be dualised to a scalar, in 3D.
- Combining the scalar with $\Box\Box \longrightarrow$ traceful $h_{ab} \sim \Box\Box \in Gl(3)$
- Traceless rank-3 with remaing vector $\longrightarrow$ traceful $\varphi_{abc} \sim \Box\Box\Box \in Gl(3)$

- Final spectrum : $\{ \varphi_{abc} , h_{ab} \}$.

- Gauge transformations :

$$\begin{cases} \delta \varphi_{abc} = 3\, \partial_{(a} \hat{\xi}_{bc)} - \frac{2}{3}\, \varepsilon_{(a}{}^{pq}\, \eta_{bc)}\, \partial_p \epsilon_q \,, \\ \delta h_{ab} = 2\, \partial_{(a} \epsilon_{b)} + 2\, \varepsilon_{pq(a}\, \partial^p \hat{\xi}^q{}_{b)} \,, \end{cases}$$

- Action :

$$S[\varphi_{abc}, h_{ab}] = \frac{1}{2} \int d^3x \Big[-\partial_a \varphi_{bcd}\, \partial^a \varphi^{bcd} + \partial^a \varphi^b\, \partial^c \varphi_{abc} + \partial_a \varphi^{abc}\, \partial^d \varphi_{bcd} - \frac{1}{7}\, \partial_a \varphi_b\, \partial^a \varphi^b - \frac{31}{28}\, \partial_a \varphi^a\, \partial^b \varphi_b$$

$$+ \frac{1}{2}\, \partial_a h_{bc}\, \partial^a h^{bc} + \frac{1}{14}\, \partial_a h\, \partial^a h - \frac{3}{7}\, \partial^a h_{ab}\, \partial_c h^{bc} - \frac{1}{7}\, \partial^a h\, \partial_c h_a{}^c$$

$$+ \frac{10}{7}\, \varepsilon_{apq}\, \partial^b h_b{}^a\, \partial^p \varphi^q - 2\, \varepsilon_{apq}\, \partial^b h^{ac}\, \partial^p \varphi^q{}_{bc} \Big]$$

- Gauge transformations entangled
- "Wrong" relative kinetic term

2.4) Relation with the triplet spin-2

The action $S[h_{ab}, Z_a]$ derived above is a member of the 1-parameter family

$$S[h_{ab}, A_a] = \frac{1}{2}\int d^3x \,\Big[-\partial_a h_{bc}\,\partial^a h^{bc} + (\alpha+2)\,\partial\cdot h_a\,\partial\cdot h^a - (\alpha+1)\,\partial^a h\,(2\,\partial\cdot h_a - \partial_a h)$$
$$- \frac{\alpha}{2}\,F_{ab}(A)\,F^{ab}(A) - \alpha\,\varepsilon_{abc}\,\partial\cdot h^a\,F^{bc}(A)\Big]$$

invariant under

$$\delta h_{ab} = 2\,\partial_{(a}\epsilon_{b)} \qquad , \qquad \delta A_a = \partial_a\lambda + \varepsilon_{abc}\,\partial^b\epsilon^c \;.$$

The case obtained by off-shell dualisation corresponds to $\alpha = -1$.

. As above, one dualises the vector $A_a \to \varphi$ scalar in 3D :

$$S[h_{ab}, \varphi] = \frac{1}{2}\int d^3x\,\Big[-\partial_a h_{bc}\,\partial^a h^{bc} + 2\,\partial\cdot h_a\,\partial\cdot h^a - (\alpha+1)\,\partial^a h\,(2\,\partial\cdot h_a - \partial_a h)$$
$$- 4\alpha\,(\partial_a\varphi\,\partial^a\varphi - \varphi\,\partial^a\partial^b h_{ab})\Big]$$

$$S[h_{ab},\varphi] = \frac{1}{2}\int d^3x\,[-\partial_a h_{bc}\,\partial^a h^{bc} + 2\,\partial\cdot h_a\,\partial\cdot h^a - (\alpha+1)\,\partial^a h\,(2\,\partial\cdot h_a - \partial_a h)$$

$$-4\alpha\,(\partial_a\varphi\,\partial^a\varphi - \varphi\,\partial^a\partial^b h_{ab})] \qquad (*)$$

Invariant under $\quad \delta h_{ab} = 2\,\partial_{(a}\epsilon_{b)} \quad , \quad \delta\varphi = -\partial^a\epsilon_a$

This is equivalent, when $\alpha = 1$, to the triplet system

$$S[h_{ab}, C_a, D] = \int d^n x\,[-\tfrac{1}{2}\,\partial_a h_{bc}\,\partial^a h^{bc} + 2\,C^a\,\partial\cdot h_a + 2\,\partial\cdot C\,D + \partial_a D\,\partial^a D - C^a C_a]$$

Invariant under $\quad \delta h_{ab} = 2\,\partial_{(a}\epsilon_{b)} \quad , \quad \delta C_a = \Box\epsilon_a \quad , \quad \delta D = \partial\cdot\epsilon$

The field C_a is auxiliary, its e.o.m. give $\quad C_a = \partial\cdot h_a - \partial_a D$.

Substituting inside the triplet action $S[h_{ab}, C_a, D]$ reproduces the action $(*)$

for $\quad n = 3 \quad , \quad \alpha = -1 \quad$ and $\quad D = -\varphi$

2.5) Spin 3 / Spin 2 system

The action shown above

$$S[\varphi_{abc}, h_{ab}] = \frac{1}{2}\int d^3x \Big[-\partial_a \varphi_{bcd}\, \partial^a \varphi^{bcd} + \partial^a \varphi^b\, \partial^c \varphi_{abc} + \partial_a \varphi^{abc}\, \partial^d \varphi_{bcd} - \frac{1}{7}\, \partial_a \varphi_b\, \partial^a \varphi^b - \frac{31}{28}\, \partial_a \varphi^a\, \partial^b \varphi_b$$

$$+ \frac{1}{2}\, \partial_a h_{bc}\, \partial^a h^{bc} + \frac{1}{14}\, \partial_a h\, \partial^a h - \frac{3}{7}\, \partial^a h_{ab}\, \partial_c h^{bc} - \frac{1}{7}\, \partial^a h\, \partial_c h_a{}^c$$

$$+ \frac{10}{7}\, \varepsilon_{apq}\, \partial^b h_b{}^a\, \partial^p \varphi^q - 2\, \varepsilon_{apq}\, \partial^b h^{ac}\, \partial^p \varphi^q{}_{bc} \Big]$$

invariant under $\delta \varphi_{abc} = 3\, \partial_{(a} \hat{\xi}_{bc)} - \frac{2}{3}\, \varepsilon_{(a}{}^{pq}\, \eta_{bc)}\, \partial_p \epsilon_q \quad , \quad \delta h_{ab} = 2\, \partial_{(a} \epsilon_{b)} + 2\, \varepsilon_{pq(a}\, \partial^p \hat{\xi}^q{}_{b)}$

is a member of the family

$$S[\varphi_{abc}, h_{ab}] = \frac{1}{2}\int d^3x \Big[a_0\, \partial_a \varphi_{bcd}\, \partial^a \varphi^{bcd} + a_1\, \partial^a \varphi^b\, \partial^c \varphi_{abc} + a_2\, \partial_a \varphi^{abc}\, \partial^d \varphi_{bcd} + a_3\, \partial_a \varphi_b\, \partial^a \varphi^b + a_4\, \partial_a \varphi^a\, \partial^b \varphi_b$$

$$+ b_0\, \partial_a h_{bc}\, \partial^a h^{bc} + b_1\, \partial_a h\, \partial^a h + b_2\, \partial^a h_{ab}\, \partial_c h^{bc} + b_3\, \partial^a h\, \partial_c h_a{}^c$$

$$+ c_1\, \varepsilon_{apq}\, \partial^b h_b{}^a\, \partial^p \varphi^q + c_2\, \varepsilon_{apq}\, \partial^b h^{ac}\, \partial^p \varphi^q{}_{bc} \Big]$$

invariant under

$$\delta \varphi_{abc} = 3\, \partial_{(a} \hat{\xi}_{bc)} - 3\, x\, \varepsilon_{(a}{}^{pq}\, \eta_{bc)}\, \partial_p \epsilon_q \quad , \quad \delta h_{ab} = 2\, \partial_{(a} \epsilon_{b)} - 2\, z\, \varepsilon_{pq(a}\, \partial^p \hat{\xi}^q{}_{b)}$$

- We find that $z = 0$ iff $x = 0$.

In that case, $c_1 = 0 = c_2$.

The action is the sum or difference of Fronsdal and Fierz-Pauli actions

↳ We reject that case, hence $x \neq 0$ and $z \neq 0$

- For the traceless part $\hat{\varphi}_{abc}$ to appear in the action,

a_0, a_1, a_2 and c_2 cannot all vanish

and as a result we find the $b_0 \neq 0$.

- Finaly, the parameters are fixed uniquely in terms of z and $\gamma = \mathrm{sign}(b_0)$

  in the case $a_0 = -1$ that one reaches by rescaling φ_{abc}.

One parameter and one sign : $\quad x = -\dfrac{2\gamma z}{9(3\gamma z^2 - 2)} \quad , \qquad \gamma = \pm 1 \quad .$

For $\gamma = +1$, $z = \pm\sqrt{\frac{2}{3}}$ rejected since it amounts to removing the piece $\partial_{(a}\epsilon_{b)}$ in δh_{ab}.

Rem. :

- There is an isolated point where $a_0 = 0$.

  By rescaling φ_{abc} so that $a_2 = -1$, all the other coefficients are fixed uniquely.

- One shows that these systems are not equivalent to any known indecomposable (triplet-like) systems.

- The original system obtained by higher dualisation of the Fierz-Pauli action is recovered for

$$\gamma = +1 \;,\quad z = -1 \quad \Rightarrow \quad x = \frac{2}{9} \;.$$

3 First-order reformulation of the spin 2 - spin 3 systems & Deformations

3.1) In flat space

A theorem [M. Grigoriev, K. Mkrtchyan, E. Skvortsov 2005] states that all topological systems in 3D are equivalent to Chern-Simons-like models.

We find that, with the Lorentz-valued one-forms $(e^a, \omega^a, E^{aa}, \Omega^{aa})$, the action

$$S = \int_{M_3} \left[\omega_a \left(de^a - \tfrac{1}{2} \varepsilon^{apq} h_p \omega_q \right) + 2z\, \omega_a h_b \Omega^{ab} + \frac{2z}{3x} \Omega_{aa} \left(dE^{aa} + \tfrac{2}{3} \varepsilon^{abc} h_b \Omega_c{}^a \right) \right]$$

$$x = -\frac{2\gamma z}{9(3\gamma z^2 - 2)}$$

invariant under

$$\delta e^a = d\epsilon^a - \varepsilon^{abc} h_b \tilde{\lambda}_c + 2z\, h_b \tilde{\alpha}^{ab} , \qquad \delta \omega^a = d\tilde{\lambda}^a ,$$

$$\delta E^{aa} = d\xi^{aa} + \tfrac{4}{3} h_b \varepsilon^{abc} \tilde{\alpha}_c{}^a - 3x \left(h^a \tilde{\lambda}^a - \tfrac{1}{3} \eta^{aa} h^b \tilde{\lambda}_b \right) , \qquad \delta \Omega^{aa} = d\tilde{\alpha}^{aa}$$

• This action exactly reproduces the 2nd order action $S[h_{ab}, \varphi_{abc}]$

presented above upon (i) eliminating the auxiliary fields (ω^a, Ω^{aa}).

(ii) fixing the Lorentz gauges where $e_{[a,b]} \overset{(*)}{=} 0$ & $E_{a,bc}\Big|_{\widehat{\begin{smallmatrix} b & c \\ a & \end{smallmatrix}}} \overset{(*)}{=} 0$

• Recall

$$\delta e_{a,b} = \partial_a \epsilon_b - \Lambda_{ab} + 2z\,\tilde{\alpha}_{ab}$$

$$\delta E_{a,bc} = \partial_a \xi_{bc} - \alpha_{bc,a} + \varkappa\left(\eta_{bc}\tilde{\Lambda}_a - 3\eta_{a(b}\tilde{\Lambda}_{c)}\right)$$

where

$$\tilde{\alpha}_{ab} := \frac{1}{2}\varepsilon_{apq}\,\alpha_b{}^{p,q} \quad , \quad \tilde{\Lambda}_a := \frac{1}{2}\varepsilon_{abc}\Lambda^{bc} \quad , \quad \alpha_{bc,a} \sim \widehat{\boxed{\begin{smallmatrix} b & c \\ a & \end{smallmatrix}}}$$

• The residual gauge transformations are $\alpha^{res.}_{a,bc} = \partial_a \xi_{bc}\Big|_{\widehat{\begin{smallmatrix} b & c \\ a & \end{smallmatrix}}}$, $\Lambda^{res.}_{ab} = \partial_{[a}\epsilon_{b]}$

In this gauge, the fields $\varphi_{abc} := 3E_{(a,bc)}$ and $h_{ab} = 2e_{(a,b)}$ transform as

$$\delta\varphi_{abc} = 3\,\partial_{(a}\hat{\xi}_{bc)} - 3\varkappa\,\varepsilon_{(a}{}^{pq}\eta_{bc)}\partial_p\epsilon_q \quad , \quad \delta h_{ab} = 2\,\partial_{(a}\epsilon_{b)} - 2z\,\varepsilon_{pq(a}\partial^p\hat{\xi}^q{}_{b)}$$

- One can express the action as

$$S = \int_{M_3} \left[\omega_a R^a(e) + e_a R^a(\omega) + \frac{2z}{3x}\left(\Omega_{ab} R^{ab}(E) + E_{ab} R^{ab}(\Omega) \right) \right] \qquad (*)$$

$$x = -\frac{2\gamma z}{9(3\gamma z^2 - 2)}$$

where $R^a(e) := de^a - \varepsilon^{apq} h_p \omega_q + 2z\, h_b \Omega^{ab}$, $R^a(\omega) := d\omega^a$,

$R^{aa}(E) := dE^{aa} + \frac{4}{3} h_p \varepsilon^{pqa} \Omega_q{}^a - 3x\left(h^a \omega^a - \frac{1}{3} \eta^{aa} h^b \omega_b \right)$, $R^{aa}(\Omega) = d\Omega^{aa}$.

3.2) There is no non-Abelian extension (alas)

Search for $S[e^a, \omega^a, E^{ab}, \Omega^{ab}] = \int_{M_3} \mathrm{Tr}\left(A\, dA + \frac{1}{3} A^3 \right)$,

- $A = \omega^a J_a + (h^a + e^a) P_a + \frac{1}{2} \Omega^{ab} J_{ab} + E^{ab} P_{ab}$,

- $\mathrm{Tr}(J_a P_b) = \eta_{ab}$
- $\mathrm{Tr}(J_{ab} P_{cd}) = \frac{z}{3x}\left(\eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc} - \frac{2}{3} \eta_{ab}\eta_{cd} \right)$.

Initial data for a possible non-Abelian algebra: $[P_a, J_b] = -\varepsilon_{abc} P^c - 3x\, P_{ab}$, $[P_a, P_b] = 0 = [P_a, P_{bc}]$,

$$[P_a, J_{bc}] = 2z\left(\eta_{a(b} P_{c)} - \frac{1}{3} \eta_{bc} P_a \right) + \frac{4}{3} \varepsilon_{a(b}{}^m P_{c)m}$$

General Ansatz for the other commutators $\longrightarrow$ No solution for Jacobi identities (20 identities to be checked)

3.3) Extension to (A)dS_3

$$\nabla^2 V^a = -\sigma\, h^a h_b V^b \quad , \quad \sigma = +1 \text{ for } AdS_3 \quad , \quad \sigma = -1 \text{ for } dS_3 \, .$$

We can keep the action in the same form (*) as above

$$S = \int_{M_3} \left[\omega_a R^a(e) + e_a R^a(\omega) + \frac{2z}{3x} \left(\Omega_{ab} R^{ab}(E) + E_{ab} R^{ab}(\Omega) \right) \right] \qquad (*)$$

for the deformed curvatures

$$\begin{aligned}
R^a(e) &= \nabla e^a + \lambda x_1\, \varepsilon^{abc}\, h_b\, e_c + x_2 \varepsilon^{abc}\, h_b\, \omega_c + \lambda x_3\, h_b E^{ab} + x_4\, h_b\, \Omega^{ab} \,, \\
R^a(\omega) &= \nabla \omega^a + \lambda^2 x_5\, \varepsilon^{abc} h_b e_c + \lambda x_6\, \varepsilon^{abc}\, h_b\, \omega_c + \lambda^2 x_7\, h_b\, E^{ab} + \lambda x_8\, h_b\, \Omega^{ab} \,, \\
R^{aa}(E) &= \nabla E^{aa} + \lambda x_9 \left(h^a e^a - \tfrac{1}{3}\, \eta^{aa}\, h^b\, e_b \right) + x_{10} \left(h^a \omega^a - \tfrac{1}{3}\, \eta^{aa}\, h^b\, \omega_b \right) \\
&\quad + \lambda x_{11}\, h_b\, \varepsilon^{abc}\, E_c{}^a + x_{12}\, h_b\, \varepsilon^{abc}\, \Omega_c{}^a \,, \\
R^{aa}(\Omega) &= \nabla \Omega^{aa} + \lambda^2 x_{13} \left(h^a e^a - \tfrac{1}{3}\, \eta^{aa}\, h^b\, e_b \right) + \lambda x_{14} \left(h^a \omega^a - \tfrac{1}{3}\, \eta^{aa}\, h^b\, \omega_b \right) \\
&\quad + \lambda^2 x_{15}\, h_b\, \varepsilon^{abc}\, E_c{}^a + \lambda x_{16}\, h_b\, \varepsilon^{abc}\, \Omega_c{}^a \,.
\end{aligned}$$

The corresponding gauge transformations being

$$\boxed{\begin{aligned}
\delta\begin{pmatrix}\lambda e^a\\ \omega^a\end{pmatrix} &= \nabla\begin{pmatrix}\lambda\xi^a\\ \tilde{\Lambda}^a\end{pmatrix} + \lambda A\,\varepsilon^{abc}h_b\begin{pmatrix}\lambda\xi_c\\ \tilde{\Lambda}_c\end{pmatrix} + \lambda B\,h_b\begin{pmatrix}\lambda\xi^{ab}\\ \tilde{\alpha}^{ab}\end{pmatrix}\,,\\
\delta\begin{pmatrix}\lambda E^{aa}\\ \Omega^{aa}\end{pmatrix} &= \nabla\begin{pmatrix}\lambda\xi^{aa}\\ \tilde{\alpha}^{aa}\end{pmatrix} + \lambda C\,\left(h^a\delta^a_b - \tfrac{1}{3}\eta^{aa}h_b\right)\begin{pmatrix}\lambda\xi^b\\ \tilde{\Lambda}^b\end{pmatrix} + \lambda D\,\varepsilon^{abc}h_b\begin{pmatrix}\lambda\xi_c{}^a\\ \tilde{\alpha}_c{}^a\end{pmatrix}\,,
\end{aligned}}$$

with the numerical matrices explicitly given by

$$A = \begin{pmatrix}x_1 & x_2\\ x_5 & x_6\end{pmatrix}\,,\quad B = \begin{pmatrix}x_3 & x_4\\ x_7 & x_8\end{pmatrix}\,,\quad C = \begin{pmatrix}x_9 & x_{10}\\ x_{13} & x_{14}\end{pmatrix}\,,\quad D = \begin{pmatrix}x_{11} & x_{12}\\ x_{15} & x_{16}\end{pmatrix}\,.$$

The requirement of gauge invariance of the curvatures gives sixteen quadratic equations on the parameters x_i. In matrix form, they read

$$\boxed{\begin{aligned}
A^2 - \tfrac{5}{6}\,BC &= \sigma I\,,\\
\tfrac{1}{2}\,D^2 + CB &= 2\sigma I\,,\\
-\tfrac{3}{2}\,DC + CA &= 0\,,\\
-\tfrac{3}{2}\,BD + AB &= 0\,.
\end{aligned}}$$

A particularly simple solution is given by

$$A = \begin{pmatrix} 0 & 1 \\ \frac{9\sigma}{4} & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 0 & 1 \\ \frac{3\sigma}{2} & 0 \end{pmatrix} = C , \quad D = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix} .$$

(It's not the only one)

Field redefinitions and flat limit

The redefined fields $\begin{pmatrix} \lambda e^a \\ \omega^a \end{pmatrix} = M \begin{pmatrix} \lambda e'^a \\ \omega'^a \end{pmatrix} , \quad \begin{pmatrix} \lambda E^{aa} \\ \Omega^{aa} \end{pmatrix} = N \begin{pmatrix} \lambda E'^{aa} \\ \Omega'^{aa} \end{pmatrix}$

with redefined gauge parameters $\begin{pmatrix} \lambda \xi^a \\ \tilde{\Lambda}^a \end{pmatrix} = M \begin{pmatrix} \lambda \xi'^a \\ \tilde{\Lambda}'^a \end{pmatrix} , \quad \begin{pmatrix} \lambda \xi^{aa} \\ \tilde{\alpha}^{aa} \end{pmatrix} = N \begin{pmatrix} \lambda \xi'^{aa} \\ \tilde{\alpha}'^{aa} \end{pmatrix}$

where $M , N \in GL(2,\mathbb{R})$

will be related by the same transformation laws,

with $A' = M^{-1} A M$, $B' = M^{-1} B N$, $C' = N^{-1} C M$, $D' = N^{-1} D N$

$\hookrightarrow$ Two solutions of $\boxed{}$ related by $Gl(2,\mathbb{R}) \times Gl(2,\mathbb{R})$ are regarded as equivalent.

Also, if (A,B,C,D) is a solution, then so is $(-A,-B,-C,-D)$.

Classification of the solutions

- Using $GL(2,\mathbb{R}) \times GL(2,\mathbb{R})$, the matrices (A, B, C, D) can be brought to one of

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \mu & -\nu \\ \nu & \mu \end{pmatrix} \qquad \text{all in } Mat(2,\mathbb{R}).$$

- A detailed analysis shows that there are two general classes of solutions:

1. The 4 matrices are real diagonal. This requires $\sigma = 1$, AdS_3 background

2. The 4 matrices are all antisymmetric ($\mu = 0$).

   ↳ This requires $\sigma = -1$, dS_3 background

In the flat case where formally $\sigma = 0$, we recover our previous results.

1. When the 4 matrices are diagonal, the fields (e^a, E^{aa}) and (ω^a, Ω^{aa})

form two decoupled systems, each noted (f^a, F^{aa}), with gauge

transformations
$$\begin{cases} \delta f^a = \nabla \epsilon^a + \lambda\, a\, \varepsilon^{abc} h_b \epsilon_c + \lambda\, b\, h_b \epsilon^{ab}, \\ \delta F^{aa} = \nabla \epsilon^{aa} + \lambda\, c\, (h^a \delta^a_b - \tfrac{1}{3} \eta^{aa} h_b) \epsilon^b + \lambda\, d\, \varepsilon^{abc} h_b \epsilon_c{}^a, \end{cases}$$

$(a, b, c, d) \in \mathbb{R}^4$ constrained by
$$\boxed{\begin{array}{ll} a^2 - \tfrac{5}{6} bc = \sigma, & \tfrac{1}{2} d^2 + bc = 2\sigma, \\ -\tfrac{3}{2} dc + ca = 0, & -\tfrac{3}{2} bd + ab = 0. \end{array}}$$

There exist solutions only in AdS_3 $(\sigma = 1)$.

↪ There is a solution where the spin-2 and spin-3 sectors of the system (f^a, F^{aa})

do not mix: $a = 1$, $b = 0 = c$, $d = \pm 2$ ⇝ free limit of single $SL(3,\mathbb{R})$ CS

↪ A more interesting solution mixes the two sectors (spin 2 and spin 3): $b \neq 0$, $c \neq 0$.

$$a = \tfrac{3}{2}, \quad b = 1, \quad c = \tfrac{3}{2}, \quad d = 1$$

When the 2 systems $(f^a_{(i)}, F^{aa}_{(i)})$ $i = 1, 2$ are considered simultaneously,

(e^a, E^{aa}) & (ω^a, Ω^{aa}), we can combine the 2 solutions above.

Using the freedom $(a, b, c, d) \mapsto (-a, -b, -c, -d)$, there remain

- 2 solutions where the 2 systems mix spin 2 & spin 3 :

$$A_1 = \frac{3}{2}\begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}, \quad C_1 = \frac{3}{2}\begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}, \quad \eta = \pm 1, \quad \sigma = +1,$$

and

- 4 solutions when only one system, say (e^α, E^{aa}), mixes spin 2 & spin 3.

$$A_2 = \begin{pmatrix} 3/2 & 0 \\ 0 & \eta_1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 3/2 & 0 \\ 0 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2\eta_2 \end{pmatrix}, \quad \eta_i = \pm 1, \quad \sigma = +1 .$$

(we exclude the cases where $B = 0 = C$)

2. <u>Antisymmetric case</u> The other solutions for the system of constraints for the matrices (A, B, C, D) are

$$A_3 = \frac{3}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C_3 = \frac{3}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \sigma = -1 \rightsquigarrow dS_3 .$$

<u>Ccl</u> : 1. & 2. fully classify the solutions of the system of constraints on (A, B, C, D)

<u>Rem</u> : Using $(M, N) \in GL(2, \mathbb{R})$, the solution (A_1, B_1, C_1, D_1) with $\eta = -1$ in AdS_3 and the solution (A_3, B_3, C_3, D_3) in dS_3 can be brought in a unified form

$$A_0 = \begin{pmatrix} 0 & 1 \\ \frac{9\sigma}{4} & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & 1 \\ \frac{3\sigma}{2} & 0 \end{pmatrix} = C_0, \quad D_0 = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix} \quad \text{where } \sigma = \pm 1 .$$

[This form does not cover the case $\eta = +1$ of (A_1, B_1, C_1, D_1), or (A_2, B_2, C_2, D_2)]

Actions in (A)dS$_3$

$$S[e^a, E^{aa}, \omega^a, \Omega^{aa}] = \frac{1}{2\lambda}\int_{M_3} \left[(\lambda e_a \;\; \omega_a)\, G \begin{pmatrix} \lambda R^a(e) \\ R^a(\omega) \end{pmatrix} + (\lambda E_{aa} \;\; \Omega_{aa})\, H \begin{pmatrix} \lambda R^{aa}(E) \\ R^{aa}(\Omega) \end{pmatrix} \right]$$

where G & H non-degenerate and symmetric

- Gauge-invariance of S implies $\boxed{A^T G - G A = 0 \, , \quad G B + C^T H = 0 \, , \quad D^T H - H D = 0}$,

where $G' = M^T G M$ and $H' = N^T H N$ equivalent to (G, H) .

(A) In the solutions for (A, B, C, D) where they are all diagonal, G & H can also be taken diagonal.

A.1) When the system (f^a, F^{aa}) does not mix spin 2 & spin 3 , we get

$$S[f^a, F^{aa}] = \frac{1}{2}\int_{M_3} \left(f_a\, R^a(f) \pm F_{aa}\, R^{aa}(F) \right) \quad \text{with} \quad R^a(f) = \nabla f^a + \lambda\, \varepsilon^{abc} h_b f_c \, ,$$

$$R^{aa}(F) = \nabla F^{aa} \pm 2\lambda\, \varepsilon^{abc} h_b F_c{}^a \, .$$

A.2) When the system (f^a, F^{aa}) does mix spin 2 & spin 3, we get

$$S[f^a, F^{aa}] = \frac{1}{2}\int_{M_3} \left(f_a\, R^a(f) - \frac{2}{3}\, F_{aa}\, R^{aa}(F) \right)$$

with

$$R^a(f) = \nabla f^a + \frac{3}{2}\lambda\, \varepsilon^{abc} h_b f_c + \lambda\, h_b F^{ab},$$

$$R^{aa}(F) = \nabla F^{aa} + \frac{3}{2}\lambda \left(h^a \delta^a_b - \frac{1}{3}\eta^{aa} h_b \right) f^b + \lambda\, \varepsilon^{abc} h_b F_c{}^a .$$

- Putting the two systems $(f^a_{(i)}, F^{aa}_{(i)})$ together, we find

$$G_1 = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}, \qquad H_1 = -\frac{2}{3}\begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \qquad \text{for} \qquad (A_1, B_1, C_1, D_1)$$

where $\tau = \pm 1$ and both signs of η,

and $G_2 = \begin{pmatrix} 1 & 0 \\ 0 & \tau_1 \end{pmatrix}$, $H_2 = \begin{pmatrix} -\frac{2}{3} & 0 \\ 0 & \tau_2 \end{pmatrix}$ for (A_2, B_2, C_2, D_2), $\tau_i = \pm 1$.

Rem: Exotic kinetic terms $e\, de$ and $\omega\, d\omega$.

(B) In the case where (A_3, B_3, C_3, D_3) are antisymmetric, only in dS_3,

one uses the $GL(2,\mathbb{R}) \times GL(2,\mathbb{R})$ gauge

$$A_0 = \begin{pmatrix} 0 & 1 \\ \frac{9\sigma}{4} & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & 1 \\ \frac{3\sigma}{2} & 0 \end{pmatrix} = C_0, \quad D_0 = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix}$$

and find $G_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -H_0$. This brings $e\,R(\omega)$, $\omega\,R(e)$, $E\,R(\Omega)$ and $\Omega\,R(E)$.

Flat limits of the field equations

We showed that some solutions of the quadratic matrix-valued equations

$$\boxed{A^2 - \frac{5}{6} BC = \sigma \mathbb{1}_2 \quad , \quad D^2 + 2\, CB = 4\sigma \mathbb{1}_2 \quad , \quad CA - \frac{3}{2} DC = 0 \quad , \quad AB - \frac{3}{2} BD = 0}$$

for the 16 parameters x_i are in the $Gl(2,\mathbb{R}) \times Gl(2,\mathbb{R})$ orbit

of the following simple solution for both $\sigma = \pm 1$

$$A_0 = \begin{pmatrix} 0 & 1 \\ \frac{9\sigma}{4} & 0 \end{pmatrix} , \quad B_0 = \begin{pmatrix} 0 & 1 \\ \frac{3\sigma}{2} & 0 \end{pmatrix} = C_0 , \quad D_0 = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix} .$$

In order to recover the flat limit, must

1) act with some appropriate $(M, N)_{\gamma, z}$

2) send $\lambda \to 0$.

Matrices $(M, N)_{\gamma, z}$ are

$$M = \begin{pmatrix} \frac{3}{\sqrt{2}}\Delta & z \\ -\frac{9\sigma}{4} z & \frac{3}{\sqrt{2}}\Delta \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 1 \\ \frac{3\sigma}{4} & 0 \end{pmatrix},$$

where Δ is the square root

$$\Delta = \sqrt{\gamma\sigma\,(2\gamma z^2 - 1)}.$$

Because the numerical matrices (A, B, C, D) are real and $\gamma = \pm 1$,

extensions from flat to $(A)dS_3$ depend on values of z and γ.

- If $\gamma = +1$, we have $\Delta = \sqrt{\sigma z^2\,(2z^2 - 1)}$: the model can be extended to dS when $z^2 < 1/2$, to AdS for $z^2 > 1/2$, and to both when $z^2 = 1/2$. In particular, the original action of [1] corresponds to $z = -1$ and therefore can only be continued to AdS, not to dS.
- If $\gamma = -1$, we have $\Delta = \sqrt{\sigma z^2\,(2z^2 + 1)}$: these models can only be deformed to AdS.

Flat limits of the actions

- $$S[e^a, \omega^a, E^{aa}, \Omega^{aa}] = \frac{1}{2\lambda}\int_{(A)dS_3} \left[(\lambda e_a \quad \omega_a)\, G \begin{pmatrix} \lambda R^a(e) \\ R^a(\omega) \end{pmatrix} + (\lambda E_{aa}, \quad \Omega_{aa})\, H \begin{pmatrix} \lambda R^{aa}(E) \\ R^{aa}(\Omega) \end{pmatrix} \right]$$

The terms $e\,R(\omega)$, $\omega\,R(e)$, $E\,R(\Omega)$ and $\Omega\,R(E)$ come with λ^0.

The terms $\omega\,R(\omega)$ and $\Omega\,R(\Omega)$ come with λ^{-1} and diverge when $\lambda \to 0$

$\Rightarrow$ should set $G_{22} = 0 = H_{22}$.

The terms $e\,R(e)$ and $E\,R(E) \longrightarrow 0$ when $\lambda \to 0$.

To recover $$S = \int_{M_3} \left[\omega_a R^a(e) + e_a R^a(\omega) + \frac{2z}{3x}\left(\Omega_{ab} R^{ab}(E) + E_{ab} R^{ab}(\Omega)\right)\right] \qquad (*)$$

$$x = -\frac{2\gamma z}{9(3\gamma z^2 - 2)}$$

in the flat limit, one should have $G_{12} = G_{21} = 1$, $H_{12} = H_{21} = \frac{2z}{3x}$,

on top of the constraints on (G, H) from gauge invariance.

- Remarkably, this selects only the following values for xz :

$$xz \in \left\{ -\frac{2}{3}\, , -\frac{2}{15}\, , \frac{2}{45}\, , \frac{2}{9} \right\} .$$

Of those values, only $xz = \frac{2}{9}$ is possible for both signs of σ.

The other 3 values of xz admit $\sigma = 1$ only, i.e. AdS_3.

- The higher dual of Fierz-Pauli has $xz = -\frac{2}{9}$, hence cannot be deformed to $(A)dS_3$.

- The value $xz = \frac{2}{9}$ can be reached from the case $(A_0, B_0, C_0, D_0, G_0, H_0)$ by

acting with $\quad M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\, , \quad N = -z \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix}$

The other 3 values for xz are in the orbit of the hybrid system

$$A_2 = \begin{pmatrix} 3/2 & 0 \\ 0 & \eta_1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 3/2 & 0 \\ 0 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2\eta_2 \end{pmatrix}, \quad \eta_i = \pm 1, \quad \sigma = +1 .$$

$$G_2 = \begin{pmatrix} 1 & 0 \\ 0 & \tau_1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} -2/3 & 0 \\ 0 & \tau_2 \end{pmatrix} \quad \text{with } \tau_1 = -1 \text{ and } \tau_2 = +1 .$$

Conclusion : The one-parameter family of actions in flat space only admits deformations to (A)dS_3 for critical values of the product xz,

$$x = -\frac{2\gamma z}{9(3\gamma z^2 - 2)} .$$

④ Relation with quivers

Use spinor notation instead of vector $so(1,2)$. The one-forms can be grouped in

$$\Phi(y,x) = \sum_{N,i} \frac{1}{N!} y_{\alpha_1} \cdots y_{\alpha_N} \Phi_i^{\alpha_1 \cdots \alpha_N}(x)$$

$(e^{\alpha\alpha}, \omega^{\alpha\alpha}, E^{\alpha(4)}, \Omega^{\alpha(4)})$ in our case.

Field equations $\boxed{\nabla\Phi = Q\,\Phi}$,

where

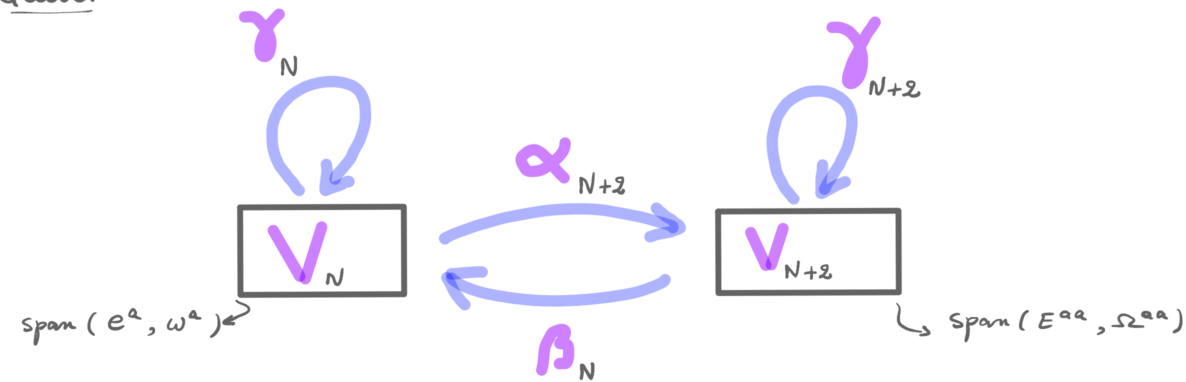
$$\boxed{Q = \alpha(N)\, h^{\alpha\alpha} y_\alpha y_\alpha + \beta(N)\, h^{\alpha\alpha} \partial_\alpha \partial_\alpha + \gamma(N)\, h^{\alpha\alpha} y_\alpha \partial_\alpha}$$

For a topological system, $\mathcal{D} = \nabla - Q$ is nilpotent. This imposes constraints on

$$\alpha_N : V_{N-2} \to V_N \,, \quad \beta_N : V_{N+2} \to V_N \,, \quad \gamma_N : V_N \to V_N$$

Moreover, redefinitions $\Phi_N \to A_N \Phi_N$ by automorphisms.

Quiver



In AdS, one can use $Gl_2 \times Gl_2$

to reach $\alpha_{N+2} = \beta_N = \gamma_N = \gamma_{N+2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It corresponds to (A_0, B_0, C_0, D_0).

In flat space, $\beta_N = \gamma_N = \gamma_{N+2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\alpha_{N+2} = q \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ 1-parameter.

- Constraints from nilpotency of $\mathcal{D}$.

Conclusions

- Strange higher-spin systems in flat space are associated with wild quivers .

- Classification of finite-dimensional representations of Poincaré algebra unknown (to us)

- Interactions were studied in the simplest spin 2 - spin 3 system