Dispersion phenomena and applications to nonlinear evolution equations

Hajer Bahouri

CNRS, LJLL, Sorbonne Université

Séminaire Orléans

Introduction

• Main goal : to investigate dispersion phenomena for linear evolution equations with applications to some nonlinear PDEs involved in physics, fluid and quantum mechanics,...on $\mathbb{R}^d$ as well as on non commutative Lie groups, and to introduce alternatives in the case of lack of dispersion

The waves decrease and vanish as the time goes to infinity

- Among the iconic examples of linear evolution equations on $\mathbb{R}^d$, one can mention

- the heat equation : $\partial_t u - \Delta u = 0$

- the transport equation : $\partial_t u + b \cdot \nabla u = 0$

- the Schrödinger equation : $i\partial_t u - \Delta u = 0$

- the wave equation : $\partial_t^2 u - \Delta u = 0$

- One can explicitly solve all these equations which are of different types

Dispersion phenomena express that waves with different frequencies move at different velocities

For instance, for the free linear Schrödinger equation on $\mathbb{R}^d$

$$(S) \qquad \begin{cases} i\partial_t u - \Delta u &= 0 \\ u_{|t=0} &= u_0 \,, \end{cases}$$

taking the partial Fourier transform $\mathcal{F}$ of (S) with respect to the variable x, we obtain

$$i\partial_t \mathcal{F}u(\xi) + |\xi|^2 \mathcal{F}u(\xi) = 0.$$

Then integrating in time the resulting ODE, we get

$$\mathcal{F}u(t,\xi) = e^{it|\xi|^2}\mathcal{F}(u_0)(\xi).$$

Applying the inverse Fourier formula, we obtain (oscillating integral)

$$u(t,x) = (2\pi)^{-d}\int e^{i(x\cdot\xi + \mathbf{t}|\xi|^{\mathbf{2}})}\mathcal{F}(u_0)(\xi)d\xi$$

$$= \Big(\frac{e^{i\frac{|\cdot|^2}{4t}}}{(4\pi i t)^{\frac{d}{2}}} \star u_0\Big)(x)$$

Commonly dispersive estimates correspond to a pointwise inequality in time decay, namely ($t \neq 0$)

$$\|u(t,\cdot)\|_{L^\infty} \lesssim \frac{\|u_0\|_{L^1}}{|t|^r}$$

where (in general) the rate of decay $r > 0$ depends on the equation, the dimension and the setting

Very often, interpolating such type of estimate with some conservation law, we deduce a family of dispersive inequalities

In the particular case of the free linear Schrödinger equation on $\mathbb{R}^d$, taking advantage of the representation of the solution under convolution form, we deduce thanks to Young inequality

$$\|u(t,\cdot)\|_{L^\infty} \leq \frac{\|u_0\|_{L^1}}{(4\pi|t|)^{\frac{d}{2}}}$$

The rate of decay $r = d/2$ is optimal thanks to the stationary phase theorem

The heart of the matter in the case of the Schrödinger equation on $\mathbb{R}^d$ is the following convolution representation

$$u(t,x) = \frac{\mathrm{e}^{-i\frac{|x|^2}{4t}}}{(-4\pi i t)^{\frac{d}{2}}} \star u_0$$

Since $\mathcal{F}u(t,\xi) = e^{it|\xi|^2}\mathcal{F}(u_0)(\xi)$, the idea consists to compute in the sense of distributions the inverse Fourier transform of the complex Gaussian

- To this end, we first take advantage of the convolution representation of the heat equation (computation of the inverse Fourier transform of the real Gaussian) to deduce that for any x in $\mathbb{R}^d$, the two following maps which are holomorphic on $\mathrm{Re}(z) > 0$ coincide

$$z \in \mathbb{C} \to H_1(z) = (2\pi)^{-d} \int \mathrm{e}^{i\langle x,\xi\rangle} \mathrm{e}^{-z|\xi|^2}\, d\xi \text{ and } z \in \mathbb{C} \to H_2(z) = \frac{1}{\left(4\pi z\right)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|x|^2}{4z}}$$

- Then, we show that if (z_p), with $\mathrm{Re}(z_p) > 0$ converges to $-it$ for $t \neq 0$, then $H_1(z_p)$ and $H_2(z_p)$ converge in $\mathcal{S}'(\mathbb{R}^d)$

Since we have the conservation of the mass for the solutions of (S) :

$$\|u(t,\cdot)\|_{L^2}^2 = \|u_0\|_{L^2}^2$$

an interpolation argument leads to the following family of dispersive inequalities, for all $1 \leq p \leq 2$

$$\|u(t,\cdot)\|_{L^p} \leq \frac{\|u_0\|_{L^{p'}}}{(4\pi|t|)^{\frac{d}{2}-\frac{d}{p}}}$$

where p' denotes the conjugate exponent of p, namely

$$\frac{1}{p} + \frac{1}{p'} = 1$$

This family of estimates is a key tool in the study of nonlinear Schrödinger equations

Actually a functional analysis argument known as the TT^*-argument initiated by Ginibre-Velo and refined by Keel-Tao enables to deduce from the family of dispersive estimates a bound for the space-time norm of the solution u by the norm of the initial datum u_0 :

$$\|u\|_{L_t^q(L_x^p)} \lesssim \|u_0\|_{L^2}$$

for some suitable (q, p) called admissible pairs

These pairs which depend on the equation and the dimension can be computed using the scale invariance of the equation

Denote $u(t,\cdot) = U(t)u_0$, then by definition of the $L^q_t(L^p_x)$-norm, we have

$$\begin{aligned}\|U(t)u_0\|_{L^q_t(L^p_x)} &= \sup_{\varphi\in\mathcal{B}_{q,p}} \Big|\int_{\mathbb{R}\times\mathbb{R}^d} U(t)u_0(x)\varphi(t,x)\,dt\,dx\Big| \\ &= \sup_{\varphi\in\mathcal{B}_{q,p}} \Big|\int_{\mathbb{R}} (U(t)u_0|\varphi(t))_{L^2}\,dt\Big|\end{aligned}$$

where $\mathcal{B}_{q,p} = \Big\{\phi \in \mathcal{D}\,/\,\|\phi\|_{L^{q'}(L^{p'})} \leq 1\Big\}$. Thus

$$\|U(t)u_0\|_{L^q_t(L^p_x)} = \sup_{\varphi\in\mathcal{B}_{q,p}} \Big|\Big(u_0\Big|\int_{\mathbb{R}} U^\star(t)\varphi(t,\cdot)\,dt\Big)_{L^2}\Big|$$

By virtue of the Cauchy-Schwarz inequality, we deduce that

$$\|U(t)u_0\|_{L^q_t(L^p_x)} \leq \|u_0\|_{L^2(\mathbb{R}^d)} \sup_{\varphi\in\mathcal{B}_{q,p}} \Big\|\int_{\mathbb{R}} U^\star(t)\varphi(t,\cdot)\,dt\Big\|_{L^2(\mathbb{R}^d)}$$

Now

$$\begin{aligned}\Big\|\int_{\mathbb{R}} U^\star(t)\varphi(t,\cdot)\,dt\Big\|^2_{L^2} &= \int_{\mathbb{R}^2}\Big(U^\star(t')\varphi(t',\cdot)\Big|U^\star(t)\varphi(t,\cdot)\Big)_{L^2}\,dt'\,dt \\ &= \int_{\mathbb{R}^2}\Big\langle U(t-t')\varphi(t',\cdot),\overline{\varphi}(t,\cdot)\Big\rangle\,dt'\,dt\end{aligned}$$

Thus by Hölder inequality

$$\Big\| \int_{\mathbb{R}} U^\star(t)\varphi(t,\cdot)\,dt \Big\|_{L^2}^2 \leq \int_{\mathbb{R}^2} \|U(t-t')\varphi(t',\cdot)\|_{L^p} \|\varphi(t,\cdot)\|_{L^{p'}}\,dt'\,dt$$

which implies thanks to the dispersive estimates

$$\Big\| \int_{\mathbb{R}} U^\star(t)\varphi(t,\cdot)\,dt \Big\|_{L^2}^2 \leq \int_{\mathbb{R}^2} \frac{1}{|t-t'|^{d(\frac{1}{2}-\frac{1}{p})}} \|\varphi(t',\cdot)\|_{L^{p'}} \|\varphi(t,\cdot)\|_{L^{p'}}\,dt'\,dt$$

This gives the result by Hardy-Littlewood-Sobolev inequalities which can be understood as a refined version of Young inequality :

$$\| |\cdot|^{-\alpha} \star f \|_{L^r(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

where $0 < \alpha < d$ and (p,r) in $]1,\infty[^2$ satisfy

$$\frac{1}{p} + \frac{\alpha}{d} = 1 + \frac{1}{r}$$

Now in the case of the Schrödinger equation on $\mathbb{R}^d$ (where the rate of decay $r = d/2$), these estimates known as Strichartz estimates are the following

$$\|u\|_{L^q(\mathbb{R}, L^p(\mathbb{R}^d))} \leq C\|u_0\|_{L^2(\mathbb{R}^d)}$$

where (q, p) are given by the scaling admissibility condition

$$\frac{2}{q} + \frac{d}{p} = \frac{d}{2}$$

and satisfy moreover $q \geq 2$ and $(d, q, p) \neq (2, 2, \infty)$.

For instance when $d = 2$, $L^3(L^6)$ is a Strichartz norm.

Note that the Strichartz estimates admit more general forms, whether in the inhomogeneous setting (with a source term f) or in more general functional spaces such as Sobolev or Besov spaces

Applications

Strichartz estimates (which express both a decrease effect and a regularity effect) constitute a central tool in the study of nonlinear equations, whether in semilinear frameworks or in quasilinear settings.

- For instance, if consider the cubic semilinear Schrödinger equation on $\mathbb{R}^2$ (which involves in quantum mechanics) :

$$(NLS_3) \qquad \begin{cases} i\partial_t u - \Delta u &= P_3(u, \overline{u}) \\ u_{|t=0} &= u_0 \in L^2(\mathbb{R}^2)\,, \end{cases}$$

where P_3 is an homogeneous polynomial of degree 3 with respect to u and $\overline{u}$ (for instance, one can take $P_3(u, \overline{u}) = |u|^2 u$), then using that $L^3_t(L^6_x)$ is a Strichartz norm (and then $\|P_3(u, \overline{u})\|_{L^1_t(L^2_x)} \sim \|u\|^3_{L^3_t(L^6_x)}$), one can show thanks to the fixed point theorem that (NLS_3) is locally (in time) wellposed for any Cauchy data in L^2 and even globally (in time) wellposed for small Cauchy data

More precisely, combining the fixed point theorem together with Strichartz estimates, one can by classical arguments establish the following theorem :

Let $u_0 \in L^2(\mathbb{R}^2)$. Then there exists a positive time $T = T(\|u_0\|_{L^2})$ such that there exists a unique solution u of (NLS_3) in the functional space $C([0,T]; L^2(\mathbb{R}^2)) \cap L^3([0,T]; L^6(\mathbb{R}^2))$. Moreover, there is a positive constant $c > 0$ such that if $\|u_0\|_{L^2} \leq c$, then the solution belongs to $L^3(\mathbb{R}_+; L^6(\mathbb{R}^2)) \cap C_b(\mathbb{R}_+; L^2(\mathbb{R}^2))$.

The defocusing (NLS_3) with $P_3(u,\overline{u}) = |u|^2 u$ is globally wellposed for any Cauchy data in $L^2(\mathbb{R}^2)$, but the proof is much more involved Benjamin Dodson,...

In the focusing case $P_3(u,\overline{u}) = -|u|^2 u$, blow up can occur for large data : wide literature Book of Cazenave

Obviously a solution of (NLS_3) is a fixed point of the following map :

$$F : u \longrightarrow F(u) = U(t)u_0 + Q(u) \quad \text{with}$$

$$i\partial_t Q(u) - \Delta Q(u) = P_3(u, \overline{u}),\ Q(u)_{|t=0} = 0$$

By Strichartz estimates, one can easily check that

$$\|F(u)\|_{L^3(\mathbb{R}_+;L^6(\mathbb{R}^2))} \lesssim \|u_0\|_{L^2(\mathbb{R}^2)} + \|u\|^3_{L^3(\mathbb{R}_+;L^6(\mathbb{R}^2))}$$

This ensures that, if $\|u_0\|_{L^2}$ is small enough, then the map F leaves invariant the ball $B(0, c\|u_0\|_{L^2})$ of $L^3(\mathbb{R}; L^6(\mathbb{R}^2))$

Moreover, one can show F is a contracting map, which gives the result thanks to the fixed point theorem

There is plethora of results in the same vein based on Strichartz estimates for nonlinear Schrödinger equations as well as for nonlinear wave equations on $\mathbb{R}^d$ but also in more general settings, such as on curved manifolds, in the presence of potentials, obstacles, boundary conditions, or in regular variable coefficient situations,...

The study of such equations which is of great importance in PDEs, has many applications in spectral theory, geometry, number theory,...

Note also that the case of quasilinear evolution equations have been also extensively studied by many authors as well as the case of bilinear estimates thanks to Strichartz estimates

Much more involved than the semilinear framework

For instance, if we consider the following wave equation in connection with the general relativity

$$(E)\begin{cases} \partial_t^2 u - \Delta u - \partial(g(u)\partial u) & = & Q(\nabla u, \nabla u) \\ (u, \partial_t u)_{|t=0} & = & (u_0, u_1) \end{cases}$$

for some suitable metric g ($g(u) \sim u$)

The basic tool to prove local solvability for such equation relies on the following energy estimate

$$\|\partial u(t,\cdot)\|_{H^{s-1}} \leq \|\partial u(0,\cdot)\|_{H^{s-1}} e^{\int_0^t \|\partial g(t',\cdot)\|_{L^\infty} dt'}$$

and the key quantity to control is then $\int_0^t \|\partial g(t',\cdot)\|_{L^\infty} dt'$

If $(\partial u_0, u_1) \in H^{s-1}$ with $s > d/2+1$, then this quantity can be controlled thanks to Sobolev embedding

To go below this regularity for the initial data, we have to combine the dispersive properties of the wave equation with the Littlewood-Paley theory and the Paradifferential calculus of J.-M. Bony

Many works bave been devoted to this equation Bahouri-Chemin, Klainerman-Rodnianski-Szeftel, Smith-Tataru, Tataru,.... it is the combination of geometrical optics and harmonic analysis that allows to get closer to the critical space.

Roughly speaking, using the paradifferential calculus of J.-M. Bony, we reduce the issue to the study of u_q the part of the solution relating to frequencies of size 2^q which satisfies a wave equation with regular coefficients, and for which we establish a microlocal Strichartz estimates, namely Strichartz estimates on time intervals whose size depend on the frequency. This is due to the fact that the regular coefficients of the wave equation satisfied by u_q take in mind the starting regularity and cannot be uniformly bounded with respect to 2^q.

To conclude, we glue the microlocal estimates to obtain a local Strichartz estimate and improve the threshold regularity given by the energy estimate

Note that for the free wave equation on $\mathbb{R}^d$

$$(W)\begin{cases}\partial_t^2 u - \Delta u = 0\\ u_{|t=0} = u_0\\ (\partial_t u)_{|t=0} = u_1\end{cases}$$

the rate of decay is $r = \frac{d-1}{2}$ and the proof requires more elaborate technics

When the Cauchy data is frequency localized in a unit ring $\mathcal{C}$,

$$u = u_+ + u_- \quad \text{with} \quad u_\pm(t,x) = (2\pi)^{-d}\int e^{i(x\cdot\xi \pm t|\xi|)}\widehat{\gamma}_\pm(\xi)d\xi\,,$$

where $\widehat{\gamma}_\pm(\xi) = \frac{1}{2}\Big(\widehat{u}_0(\xi) \mp \frac{i}{|\xi|}\widehat{u}_1(\xi)\Big)$. Thus by inversion Fourier formula,

$$u_\pm(t,\cdot) = K_\pm(t,\cdot) \star \gamma_\pm \quad \text{with}$$

$$K_\pm(t,x) = (2\pi)^{-d}\int e^{i(x\cdot\xi \pm t|\xi|)}\varphi(\xi)d\xi$$

Then by virtue of Young inequality, we get

$$\|u_\pm(t,\cdot)\|_{L^\infty} \leq \|K_\pm(t,\cdot)\|_{L^\infty}\|\gamma_\pm\|_{L^1} \lesssim \|K_\pm(t,\cdot)\|_{L^\infty}\Big(\|u_0\|_{L^1} + \|u_1\|_{L^1}\Big)$$

Thus the problem reduces to the estimate of the L^∞-norm of $K_\pm(t,\cdot)$

To this end, we apply stationary phase theorem (integration by parts with respect to a suitable vector field) to

$$K_\pm(t,tX) = (2\pi)^{-d}\int e^{it\Phi_\pm(X,\xi)}\varphi(\xi)d\xi$$

where

$$\Phi_\pm(X,\xi) = X\cdot\xi \pm |\xi|$$

The study of oscillatory integrals is in connection with the critical point of the phase, here $\Phi_\pm$, with respect to the integration variable, here ξ

General setting

Nowadays, it is well known that the geometry of the setting has a big influence on the dispersion effect, and many authors have focused on these matters. Among others, one can mention the results of

- Anker-Pierfelice, Banica, Pierfelice,... on the real hyperbolic space

- Bourgain on the Torus

- Burq-Gérard-Tzvetkov and Staffilani-Tataru on compact manifolds

- Ivanovici-Lebeau-Planchon,... on some domains

- Banica-Duyckaerts,... on noncompact manifolds

- Bahouri-Gérard-Xu, Bahouri-Fermanian-Gallagher, Del Hierro, Müller-Seeger, Furioli-Melzi-Veneruso,... on stratified Lie groups such as the Heisenberg group $\mathbb{H}^d$ which is a step 2 stratified Lie group

In particular there are many situations where dispersion phenomena fail ! (for instance in compact manifolds, on the Heisenberg group,...)

The Schrödinger equation on $\mathbb{H}^d$ ($(Y, s) \in \mathbb{R}^{2d} \times \mathbb{R}$ is a generic element of $\mathbb{H}^d$) behaves as a transport equation along the center generated by s
Bahouri-Gérard-Xu

However in some cases, Strichartz estimates (in weak forms) or smoothing properties can be established using other approches, which are not based on dispersive inequalities

Among these approches, one can mention the methods based on

- Fourier restriction theorems

- Kato smoothing effect

Fourier restriction theorems

The subject is wide and there are several references and monographs

Tao : Some recent progress on the restriction conjecture, Fourier Analysis and Convexity

Stein : Harmonic Analysis : Real-Variable Methods, Orthogonality, Oscillatory integrals

On $\mathbb{R}^d$, the basic Fourier restriction theorem which is due to Tomas-Stein states as follows :

Let $d \geq 2$ and $p_0 = \frac{2(d+1)}{d+3}$. Let S be a smooth and compact hypersurface in $\mathbb{R}^d$ endowed with a smooth measure $d\sigma$. Assume that S has **non vanishing Gaussian curvature** at every point. Then for all $p \in [1, p_0]$, there exists a positive constant $C = C_p$ such that for any $f \in \mathcal{S}(\mathbb{R}^d)$:

$$\|(\mathcal{F}f)_{|S}\|_{L^2(S,d\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^d)}$$

p_0 is optimal according to Knapp's counter-example

For the Heisenberg group $\mathbb{H}^d$, we have the following Fourier restriction theorem due to Müller : for $1 \leq p \leq 2$

$$\|\mathcal{F}_{\mathbb{H}}(f)_{|\mathbb{S}_{\widehat{\mathbb{H}}^d}}\|_{L^2(\mathbb{S}_{\widehat{\mathbb{H}}^d})} \leq C\|f\|_{L^p_Y L^1_s}$$

To be aware : contrary to $\mathbb{R}^d$ the Fourier dual of $\mathbb{H}^d$ is not $\mathbb{H}^d$!

No gain in s direction, but the gain is better than in $\mathbb{R}^{2d}$ for Y

Tomas-Stein's theorem gives an answer to the following problem : can we restrict the Fourier transform of an L^p function to a subset ?

- When $p = 1$, the answer is obvious since by Riemann-Lebesgue theorem, $\mathcal{F} : L^1(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$

- When $p = 2$, $\mathcal{F}f$ is in $L^2(\mathbb{R}^d)$ so it is arbitrary on any zero measure subset

- The Fourier transform of a L^p function, $p > 1$ cannot be always restricted to hyperplanes as shown by the following example

$$f(x) = \frac{e^{-|x'|}}{1 + |x_1|}$$

whose Fourier transform cannot be restricted to

$$\{\xi \in \mathbb{R}^d : \xi_1 = 0\}$$

Restriction theory has many applications to other topics, from number theory to PDEs :

Let us see how it provides Strichartz estimates for instance for the Schrödinger equation. We have seen that :

$$u(t,x) = (2\pi)^{-d} \int e^{i(x\cdot\xi + t|\xi|^2)} \mathcal{F}(u_0)(\xi) d\xi.$$

This formula can be interpreted as the restriction of the Fourier transform on the paraboloid S in the space of frequencies of $\mathbb{R}^{1+d}$, defined as

$$S = \left\{ (\alpha, \xi) \in \mathbb{R}^{1+d} \mid \alpha = |\xi|^2 \right\}.$$

Then (with $y = (t,x)$ and $z = (\alpha, \xi)$)

$$u(t,x) = (2\pi)^{-d} \int_S e^{iy\cdot z} g(z) d\sigma(z),$$

where $g(|\xi|^2, \xi) = \mathcal{F}(u_0)(\xi)$

Invoking the dual form of Tomas-Stein theorem, we deduce that

$$\|u\|_{L^{2+\frac{4}{d}}(\mathbb{R}, L^{2+\frac{4}{d}}(\mathbb{R}^d))} \leq C \|u_0\|_{L^2(\mathbb{R}^d)}$$

This proof is due to Strichartz

Kato smoothing effect

- This property was first established by Kato for KdV based on Fourier restriction type results, then by Constantin-Saut for systems including the Schrödinger equation (see also Ben Artzi-Devinatz, Ben Artzi-Klainerman, Sjölin, Vega, Yajima,...)

- In the case of the Schrödinger equation, even though $\|u(t,\cdot)\|_{L^2} = \|u_0\|_{L^2}$ $(\mathcal{F}u(t,\xi) = e^{it|\xi|^2}\mathcal{F}(u_0)(\xi))$ one can locally gain one half derivative in the following sense (book of Robbiano : Smoothing effects for the Schrödinger equation)

There exists a positive constant C such that, for all $u_0 \in L^2(\mathbb{R}^d)$, the solution u of the Schrödinger equation satisfies :

$$\|\langle x\rangle^{-1}\langle D_x\rangle^{\frac{1}{2}}u\|_{L^2(\mathbb{R}\times\mathbb{R}^d)} \leq C\|u_0\|_{L^2(\mathbb{R}^d)}.$$

The gain of 1/2-derivative is optimal Sun-Trélat-Zhang-Zhong

Dispersion phenomena on stratified Lie groups

Recall that one can recover the group from its Lie algebra and vice versa, by means of the formula of Baker-Campbell-Hausdorff and the exponential map

The Lie algebra $\mathfrak{g}$ of a step-r stratified Lie group admits the following stratification

$$\mathfrak{g} = \oplus_{i=1}^{r} \mathfrak{g}_i \quad \text{with} \quad \mathfrak{g}_{i+1} = [\mathfrak{g}_1, \mathfrak{g}_i]$$

where $\mathfrak{g}_1$ is Lie bracket generating. The most famous examples are

Heisenberg group (step-2) $(\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2, \overbrace{X_1, X_2}^{\mathfrak{g}_1}, \overbrace{[X_1, X_2]}^{\mathfrak{g}_2})$

Engel group G (step-3) $(\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3, \overbrace{X_1, X_2}^{\mathfrak{g}_1}, \overbrace{X_3 = [X_1, X_2]}^{\mathfrak{g}_2}, \overbrace{[X_1, X_3]}^{\mathfrak{g}_3})$

Note that on non commutative Lie groups, evolution equations involve the sublaplacian $\Delta_G = \sum_{j \in \mathfrak{g}_1} X_j^2$

The study of dispersive phenomena on stratified Lie groups is a challenging task : contrary to $\mathbb{R}^d$, the solution is not one oscillating integral, but rather a series of oscillating integrals

This is due to the fact that the Fourier transform on stratified Lie groups is an intricate tool : it is defined by a family of bounded operators on some Hilbert space

The key point relies on the **action of the group Fourier transform on the sublaplacian**. For instance

- for the Heisenberg group, the Fourier transform exchanges (the opposite) of the sublaplacian with the rescaled harmonic oscillator

$$-\Delta_z + |\lambda|^2|z|^2, \quad z \in \mathbb{R}^d, \ \lambda \in \mathbb{R}^*$$

- for the Engel group, the Fourier transform exchanges the sublaplacian with the rescaled quartic oscillator

$$-\frac{d^2}{d\theta^2} + \left(\frac{\lambda}{2}\theta^2 - \frac{\nu}{\lambda}\right)^2, \quad \theta \in \mathbb{R}, \ \lambda \in \mathbb{R}^*, \ \nu \in \mathbb{R}$$

In both cases, λ is the dual variable of the center

Then obviously the behavior of evolution equations respectively on the Heisenberg group and on the Engel group depend of the spectral analysis of the harmonic oscillator and the quartic oscillator

The spectral analysis of the harmonic oscillator is explicitly known while it is not the case the quartic oscillator (even we have a lot of available results)

Note that for general step-2 stratified Lie groups, thanks to a study of Ciatti-Ricci-Sundari, the Fourier transform exchange the sublaplacian with

$$|\nu|^2 + H(\lambda),$$

where $H(\lambda)$ is an anisotropic rescaled harmonic oscillator, where the unknown scales $\eta_j(\lambda)$ are homogeneous of degree 1 with respect to λ

Bahouri-Fermanian-Gallagher : dispersion on general step-2 stratified Lie groups

The case of general stratified Lie groups is not known, but we have some results for the filiform groups Bahouri-Barilari-Gallagher-Léautaud

The Heisenberg group

$S_{\mathbb{H}}$ is a totally non dispersive equation Bahouri-Gérard-Xu

Roughly speaking, if $u(t,\cdot)$ denotes the solution of $S_{\mathbb{H}}$ generated by u_0, then

$$(\mathcal{F}_{\mathbb{H}})u(t,\cdot)(n,m,\lambda) = e^{it4|\lambda|(2|m|+d)}(\mathcal{F}_{\mathbb{H}}u_0)(t,\cdot)(n,m,\lambda)$$

Actually $4|\lambda|(2|m|+d)$ plays the role of $|\xi|^2$

By Fourier inversion formula, we obtain a representation of the solution in terms of a series of oscillating integrals

$$u(t,Y,s) = C_d \sum_m \int_{-\infty}^{\infty} e^{is\lambda} e^{it4|\lambda|(2|m|+d)} \ldots d\lambda$$

One can deduce that $S_{\mathbb{H}}$ is a superposition of transport equation with velocity $\pm 4(2|m|+d)$ and in particular exhibit data u_0 so that

$$u(t,Y,s) = u_0(Y, s+4d)$$

However, taking advantage of Gaveau formula for the heat kernel on $\mathbb{H}^d$,

$$h_t(Y,s) = \frac{1}{(4\pi t)^{\frac{Q}{2}}} \int_{\mathbb{R}} \left(\frac{2\tau}{\sinh 2\tau}\right)^d \exp\left(i\frac{\tau s}{2t} - \frac{|Y|^2\tau}{2t\tanh 2\tau}\right) d\tau,$$

we established that the Schrödinger kernel on $\mathbb{H}^d$ satisfies

$$S_t(Y,s) = \frac{1}{(-4i\pi t)^{\frac{Q}{2}}} \int_{\mathbb{R}} \left(\frac{2\tau}{\sinh 2\tau}\right)^d \exp\left(-\frac{\tau s}{2t} - i\frac{|Y|^2\tau}{2t\tanh 2\tau}\right) d\tau\,,$$

provided that $|s| < 4d|t|$ Bahouri-Gallagher

This allows us to derive local dispersion for Schrödinger in $\mathbb{H}^d$

If $u_0 \in \mathcal{D}(B_{\mathbb{H}}(w_0, R_0))$, then for all $\kappa < \sqrt{4d}$, $2 \leq p \leq \infty$ and $|t| \geq T_{\kappa,R_0}$

$$\|u(t,\cdot)\|_{L^p(B_{\mathbb{H}}(w_0,\kappa|t|^{\frac{1}{2}}))} \lesssim \frac{1}{|t|^{\frac{Q}{2}}}^{1-\frac{2}{p}} \|u_0\|_{L^{p'}(\mathbb{H}^d)}$$

On the other hand, based on Müller Fourier restriction theorem on $\mathbb{H}^d$ and inspired by the Euclidean Fourier restriction strategy initiated by Strichartz, we established "weak Strichartz" estimates for the solutions to the Schrödinger equation on $\mathbb{H}^d$

$$\|u\|_{L^\infty_s L^q_t L^p_Y} \leq C_{p,q}\Big(\|u_0\|_{H^{\frac{Q}{2}-\frac{2}{q}-\frac{2d}{p}}(\mathbb{H}^d)} + \|i\partial_t u - \Delta_{\mathbb{H}} u\|_{L^1(\mathbb{R},H^{\frac{Q}{2}-\frac{2}{q}-\frac{2d}{p}}(\mathbb{H}^d))}\Big)$$

Bahouri-Barilari-Gallagher

No gain in s direction

Be aware, contrary to $\mathbb{R}^d$, $\mathbb{R}\times\mathbb{H}^d$ is not some Heisenberg group : we were led to prove a Fourier restriction theorem $\mathbb{R}\times\mathbb{H}^d$

Concerning the wave equation on $\mathbb{H}^d$, we have dispersive inequalities (for data frequency localized in a unit ring) of optimal rate of decay $r = 1/2$ regardless the dimension Bahouri-Gérard-Xu.

As already mentioned, the key point consists to estimate the involved kernel which turns out as a series of oscillating integrals

Taking advantage of the Heisenberg Fourier transform and the explicit spectral analysis of the harmonic oscillator, we can be reduced to investigate

$$K = C_d \sum_m I_m \quad \text{with}$$

$$I_m(t,Y,s) = \int_{-\infty}^{\infty} e^{is\lambda} e^{it\sqrt{4|\lambda|(2m+d)}} \mathcal{L}_m^{(d-1)}(2|\lambda||Y|^2) a(4|\lambda|(2m+d))|\lambda|^d \, d\lambda,$$

with $\mathcal{L}_m^{(d-1)}(2|\lambda||Y|^2) = e^{-|\lambda||Y|^2} L_m^{(d-1)}(2|\lambda||Y|^2)$ where $L_m^{(d-1)}$ stands for the Laguerre polynomial of order m and type $d-1$ and a compactly supported away from 0

The result follows from an application of the van der Corput lemma (refined stationary phase theorem in dimension 1)

iI a is compactly supported and b is a regular function satisfying $|b''(\mu)| \geq c_0 > 0$, on the support of a, then

$$\Big| \int_{\mathbb{R}} e^{itb(\mu)} a(\mu)\, d\mu \Big| \leq C_0 |t|^{-\frac{1}{2}} \int_{\mathbb{R}} |a'(\mu)|\, d\mu,$$

where C_0 is a constant depending only on c_0

to the oscillating integral $I_m(t, Y, ts)$ after the change of variables $r = 4|\lambda|(2m + d)$, taking advantage of the following key property of Laguerre functions

$$|y\partial_y \mathcal{L}_m^{(d-1)}(y)| \lesssim (m+1)^{d-1},$$

for all $y \geq 0$ and $\ell \in \mathbb{N}$

The Engel group

The Engel group G is a step-3 stratified Lie group ($\mathbb{R}^4 = (x', x_4)$ where x_4 generates the center)

As already mentioned, the Engel Fourier transform exchanges the sublaplacian with the quartic oscillator rescaled

$$\mathsf{P}_\mu = -\frac{d^2}{d\theta^2} + \left(\frac{\theta^2}{2} - \mu\right)^2, \quad \mu \in \mathbb{R}$$

The behavior of the potential $\left(\frac{\theta^2}{2} - \mu\right)^2$ depends on the sign of the parameter μ :

- It admits a single well when $\mu < 0$

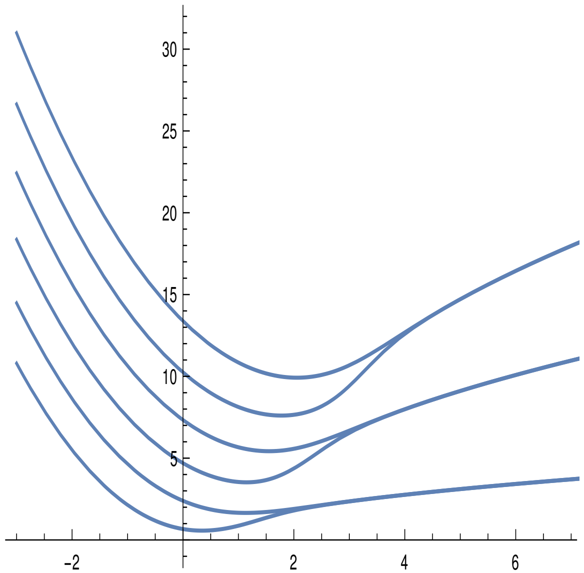
- It has a double well when $\mu > 0$

This of course influences the spectral properties of P_μ that involve in the representations of the solutions

It is well-known that

$$\mathsf{Sp}(\mathsf{P}_\mu) = \{\mathsf{E}_m(\mu), m \in \mathbb{N}\} \quad \text{with}$$

$$0 < \mathsf{E}_0(\mu) < \mathsf{E}_1(\mu) < \cdots < \mathsf{E}_m(\mu) < \mathsf{E}_{m+1}(\mu) \to +\infty,$$
$$\dim \ker(\mathsf{P}_\mu - \mathsf{E}_m(\mu)) = 1$$



Thanks to Engel Fourier analysis, the solution of (S_G) the Schrödinger equation on G writes (for some data)

$$u(t,x) = (2\pi)^{-3} \sum_m \int e^{i\lambda x_4} e^{it|\lambda|^{2/3} \mathsf{E}_m(\mu)} \mathcal{W}(\ldots) \varphi(|\lambda|^{2/3} \mathsf{E}_0(\mu)) |\lambda|^{4/3} d\lambda d\mu$$

where $\mathcal{W}(\ldots)_{|x'=0} = 1$. Then

$$u(t,0,tx_4) = (2\pi)^{-3} \sum_m \int e^{i\lambda x_4} e^{it|\lambda|^{2/3} \mathsf{E}_m(\mu)} \varphi(|\lambda|^{2/3} \mathsf{E}_0(\mu)) |\lambda|^{4/3} d\lambda d\mu$$

Since Helffer-Léautaud established that the map

$$\mu \mapsto \mathsf{E}_0(\mu)$$

admits a unique critical point $\mu_0 > 0$ which is moreover non degenerate, applying the stationary phase theorem, one can prove that (S_G) could at most disperse with a rate of decay of order $1/|t|$

Note that for $m >> 1$, Helffer-Léautaud proved the same result and showed that the critical points are asymptotically located on the curve

$$\mathsf{E}_m(\mu_m) \sim C\mu_m^2, \quad C \sim 2.35$$

In fact

$$u(t,x) = (2\pi)^{-3} \sum_m I_m \quad \text{with}$$

$$I_m(t,x) = \int_{\mu\in\mathbb{R}} \int_{0<c_1\leq r\leq c_2} e^{itr^2} \frac{r^6 \phi(r^2)}{\mathsf{E}_m(\mu)^{7/2}} A(x,\mu,r)\, d\mu dr$$

where

$$A(x,\mu,r) = e^{i\frac{r^3}{\mathsf{E}_m(\mu)^{3/2}}x_4} \int e^{i\frac{r^2}{E_m(\mu)}\theta x_3 + i\frac{r}{\sqrt{\mathsf{E}_m(\mu)}}(\frac{\theta^2}{2}-\mu)x_2} \varphi_m^\mu\Big(\theta + \frac{r}{\sqrt{\mathsf{E}_m(\mu)}}x_1\Big)\varphi_m^\mu(\theta)d\theta$$

We are able to establish weighted dispersive estimates

Work in progress Bahouri-Barilari-Gallagher-Léautaud