Decomposition of the solutions to the L^1 Monge-Kantorovitch problem on the real line

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Abstract

Given μ and ν two probability measure on $\mathbb{R}$, the set $\pi(\mu, \nu)$ of transport plans from μ to ν represents the ways to transfer mass from μ to ν. The Monge-Kantorovich problem consists of finding the transport plan(s) that minimize(s) the overall cost of mass transfer with respect to a certain cost function, i.e., minimize a functional $J : \pi \in \pi(\mu, \nu) \mapsto \int_{\mathbb{R}^2} c(x, y) d\pi(x, y)$, where $\pi(x, y)$ represents the amount of mass moved from x to y and $c(x, y)$ represents the unit cost of moving mass from x to y. In the case where $c(x, y) = |x - y|^p$ with $p > 1$ and μ admits a density, a standard result by Yann Brenier states that this problem has a unique solution given by a transport map. However, when $c(x, y) = |x - y|$, uniqueness fails and our problem has infinitely many solutions. It will be noted that that this solutions are subject to constraints (existence of barrier points and oriented mass transfer), leading to a decomposition result of transport plans. Indeed, our L^1 transport problem from μ to ν can be decomposed into several L^1 transport problems from $\tilde{\mu}$ to $\tilde{\nu}$ where the measures $\tilde{\mu}$ and $\tilde{\nu}$ are related by a stochastic order that we will describe. A "Strassen-type" result for this stochastic order will then distinguish a particular solution among the existing ones, and we will explain why this solution is selected as the limit of solutions for the entropic optimal transport (under certain assumptions about μ and ν).