Trade-Offs in Distributed Learning and Optimization

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Includes joint works with Yossi Arjevani, Nathan Srebro and Tong Zhang

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Distributed Learning and Optimization

Why?

- Big data
  - Too large to fit into a single machine
  - Distributed learning as a **constraint**
- Computational Speed-ups
  - Ideally, k machines $= 1/k$ training time
  - Distributed learning as an **opportunity**

Challenge 1: Communication

Very slow compared to local processing[1]:

L1 Cache Reference	$\sim \mathbf{0.5}$ ns
Main memory Reference	$\sim \mathbf{100}$ ns
Round-trip within same datacenter	$\sim \mathbf{500,000}$ ns
Packet California$\Rightarrow$Netherlands & back	$\sim \mathbf{150,000,000}$ ns

[1] `http://norvig.com/21-days.html#answers`

Challenges

Challenge 2: Parallelization

How to parallelize algorithms of the following form:

Challenges

Challenge 3: Accuracy

Given the same data, output quality should resemble that of a non-distributed algorithm

Setting

Convex distributed empirical risk minimization:

$$\min_{\mathbf{w}\in\mathcal{W}\subseteq\mathbb{R}^d} F(\mathbf{w}) = \frac{1}{m}\sum_{i=1}^{m} F_i(\mathbf{w})$$

$$F_i(\mathbf{w}) = \frac{1}{n}\sum_{j=1}^{n} f_{i,j}(\mathbf{w})$$

$f_{i,j}(\cdot)$: Loss on j-th example in i-th machine.

- Strongly-convex/Smooth problems: Each F_i is strongly convex/smooth

Machines can broadcast simultaneously in communication rounds

- $\tilde{\mathcal{O}}(d)$ bits per machine/round

Main Question

How to optimally trade-off

1. Accuracy: $F(\mathbf{w}) - \inf_{\mathbf{w} \in \mathcal{W}} F(\mathbf{w}) \leq \epsilon$
2. Communication
3. Runtime

Notes: In this talk, accuracy in terms of optimization error, not risk

3 Data Partition Scenarios

$$\min_{\mathbf{w}\in\mathcal{W}\subseteq\mathbb{R}^d} F(\mathbf{w}) = \frac{1}{m}\sum_{i=1}^{m} F_i(\mathbf{w}) \qquad F_i(\mathbf{w}) = \frac{1}{n}\sum_{j=1}^{n} f_{i,j}(\mathbf{w})$$

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Random partition

- mn individual losses $f_{i,j}$ assigned to machines at random
- Strong concentration of measure effects (e.g. $F_i(\cdot)$'s have similar values/gradients/Hessians)

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δ-related setting

- Generalization of previous scenarios
- Informally: Values/gradients/Hessians at any $\mathbf{w}$ are $\leq \delta$-distant across machines

Talk Outline

Algorithms + worst-case upper and lower bounds* for

1. Arbitrary partition
2. δ-related
3. Random partition
   - Relies on new results for without-replacement sampling in stochastic gradient methods

* In terms of # communication rounds

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Baseline: Reduce to non-distributed first-order optimization

Example (Gradient Descent)

- Initialize $\mathbf{w}_0$; For $t = 1, 2, \ldots$
  - Machine i computes $\nabla F_i(\mathbf{w}_t)$
  - Communication round: $\nabla F(\mathbf{w}_t) = \frac{1}{m}\sum_{i=1}^{m} \nabla F_i(\mathbf{w}_t)$
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- Can also accelerate; use proximal smoothing etc.
- $\#$ communication rounds $=$ iteration complexity
  - λ-Strongly convex $+$ 1-smooth: $\mathcal{O}\left(\sqrt{1/\lambda}\log\left(\frac{1}{\epsilon}\right)\right)$
  - λ-Strongly convex: $\mathcal{O}(\sqrt{1/\lambda\epsilon})$; Convex: $\mathcal{O}(1/\epsilon)$ (with smoothing)
- Almost full parallelization; but many comm. rounds

Arbitrary Partition

Can we improve on this simple baseline?

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Assumed Algorithmic Template

- For each machine j, let $W_j = \{\mathbf{0}\}$
- Between communication rounds: Each machine j sequentially computes and adds to W_j an arbitrary number of vectors $\mathbf{w}$, s.t. $\gamma\mathbf{w} + \nu\nabla F_j(\mathbf{w})$ (for some $\gamma, \nu \geq 0$, $\gamma + \nu > 0$) is in

$$\text{span}\Big\{\mathbf{w}'\ ,\ \nabla F_j(\mathbf{w}')\ ,\ (\nabla^2 F_j(\mathbf{w}') + D)\mathbf{w}''\ ,\ (\nabla^2 F_j(\mathbf{w}') + D)^{-1}\mathbf{w}''\ \Big|\ \mathbf{w}', \mathbf{w}'' \in W_j\ \ ,\ \ D \text{ diagonal}\Big\}$$

- During communication round: Broadcast some vectors in W_j

Proof Sketch (λ-strongly convex; 1-smooth; 2 machines)

$$
\begin{aligned}
F_1(\mathbf{w}) &= \mathbf{w}^\top \left(\frac{1-\lambda}{4} A_1 + \frac{\lambda}{2} I \right) \mathbf{w} - \frac{1-\lambda}{2} \mathbf{e}_1^\top \mathbf{w} \\
F_2(\mathbf{w}) &= \mathbf{w}^\top \left(\frac{1-\lambda}{4} A_2 + \frac{\lambda}{2} I \right) \mathbf{w}
\end{aligned}
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$$

$$
A_1 = \begin{bmatrix}
+1 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & +1 & -1 & 0 & 0 & 0 & \dots \\
0 & -1 & +1 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & +1 & -1 & 0 & \dots \\
0 & 0 & 0 & -1 & +1 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\qquad
A_2 = \begin{bmatrix}
+1 & -1 & 0 & 0 & 0 & 0 & \dots \\
-1 & +1 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & +1 & -1 & 0 & 0 & \dots \\
0 & 0 & -1 & +1 & 0 & 0 & \dots \\
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0 & 0 & 0 & 0 & -1 & +1 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
$$

- $\exp(-\Omega(T/\sqrt{1/\lambda}))$ error after T iterations
- $\frac{1}{2}F_1(\cdot) + \frac{1}{2}F_2(\cdot)$ is essentially "hard function" for *non-distributed* optimization [Nemirovski and Yudin 1983]

Proof Sketch (λ-strongly convex); 2 machines

$\Omega(1/(\lambda T^2))$ error after T iterations
(matched by AGD with proximal smoothing)

$$
\begin{aligned}
F_1(\mathbf{w}) = \frac{1}{\sqrt{2}}|b - w_1| \\
&+ \frac{1}{\sqrt{2(T+2)}}\left(|w_2 - w_3| + |w_4 - w_5| + \cdots + |w_T - w_{T+1}|\right) \\
&+ \frac{\lambda}{2}\|\mathbf{w}\|^2
\end{aligned}
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δ-related Setting

$$\min_{\mathbf{w}\in\mathcal{W}\subseteq\mathbb{R}^d} F(\mathbf{w}) = \frac{1}{m}\sum_{i=1}^{m} F_i(\mathbf{w}) \qquad F_i(\mathbf{w}) = \frac{1}{n}\sum_{j=1}^{n} f_{i,j}(\mathbf{w})$$

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- Assuming F_i are smooth, values/gradients/Hessians at any $\mathbf{w} \in \mathcal{W}$ are δ-close
- Can show similar lower bounds, but weakened by δ factors
  - e.g. $\sqrt{\frac{\delta}{\lambda}} \log\left(\frac{1}{\epsilon}\right)$ for λ-strongly convex and smooth

Can we build algorithms which utilize δ-relatedness?

More data – less communication??

DANE Algorithm

Distributed **A**pproximate **NE**wton-type method

- Parameters: Learning rate $\eta > 0$; regularizer $\mu > 0$
- Initialize $\mathbf{w}_1 = 0$
- For $t = 1, 2, \ldots$
  - Compute $\nabla F(\mathbf{w}_t) = \frac{1}{m} \sum_i \nabla F_i(\mathbf{w}_t)$
  - For each machine i, solve local optimization problem:

$$\mathbf{w}_{t+1}^i = \arg\min_{\mathbf{w}} \quad F_i(\mathbf{w}) - (\nabla F_i(\mathbf{w}_t) - \eta \nabla F(\mathbf{w}_t))^\top \mathbf{w} + \frac{\mu}{2} \|\mathbf{w} - \mathbf{w}_t\|_2^2$$

  - Compute average $\mathbf{w}_{t+1} = \frac{1}{m} \sum_i \mathbf{w}_{t+1}^i$

Structure similar to ADMM

Crucial Property

Equivalent to **approximate Newton step**

Newton step

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \left(\nabla^2 F(\mathbf{w}_t)\right)^{-1} \nabla F(\mathbf{w}_t)$$

Extremely fast convergence, but expensive

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DANE (Quadratic Functions)

Equivalent to

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \eta \left(\frac{1}{m}\sum_i \left(\nabla^2 F_i(\mathbf{w}_t)\right)^{-1} + \mu I\right) \nabla F(\mathbf{w}_t)$$

even though Hessians $\nabla^2 F_i$ never computed explicitly!

Theoretical Guarantees: Quadratic Functions

Theorem

$$\left\|\mathbf{w}_{t+1} - \mathbf{w}_{opt}\right\| \leq \left\|I - \eta \tilde{H}^{-1} H\right\|^t \left\|\mathbf{w}_1 - \mathbf{w}_{opt}\right\|$$

where

$$H = \frac{1}{m}\sum_i \nabla^2 F_i \quad , \quad \tilde{H}^{-1} = \frac{1}{m}\sum_i (\nabla^2 F_i + \mu I)^{-1}$$

- In a δ-related setting, $\left\|\nabla^2 F_i - H\right\| \leq \delta$ for all i
- Setting η, μ appropriately,

$$\left\|I - \eta \tilde{H}^{-1} H\right\| \;\leq\; \max\left\{\frac{1}{2}, 1 - \frac{1}{16(\delta/\lambda)^2}\right\}$$

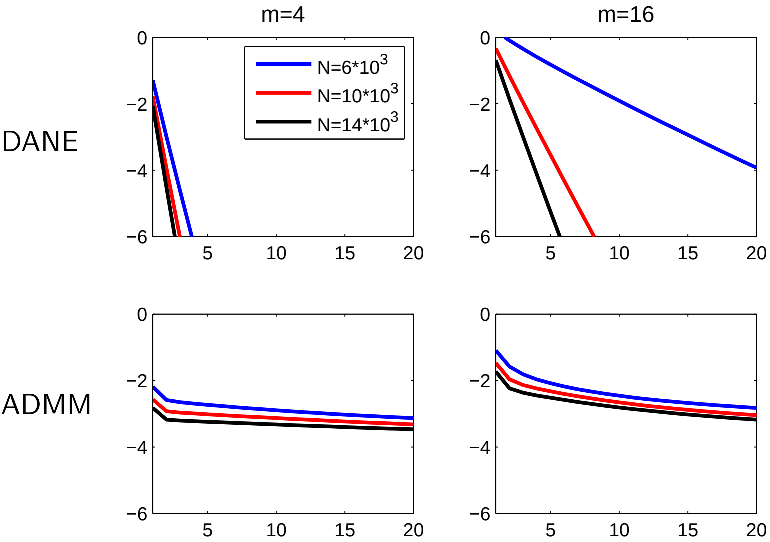
Conclusion

Need $\mathcal{O}((\delta/\lambda)^2)\log(1/\epsilon)$ communication rounds

Some Experiments

Regularized least squares in $\mathbb{R}^{500}$

$\log_{10}$(optimization error) vs. # iterations



Extensions

- Can provide some guarantees for non-quadratic losses as well
- Using a more sophisticated algorithm, can improve guarantees to $\mathcal{O}(\sqrt{\delta/\lambda}\log(1/\epsilon))$ rounds, for quadratics / self-concordant losses [Zhang and Xiao 2015]

Disadvantage: Each iteration requires solving a local optimization problem $\Rightarrow$ Not necessarily cheap in terms of runtime

Random Partition

$$\min_{\mathbf{w}\in\mathcal{W}\subseteq\mathbb{R}^d} F(\mathbf{w}) = \frac{1}{m}\sum_{i=1}^{m} F_i(\mathbf{w}) \qquad F_i(\mathbf{w}) = \frac{1}{n}\sum_{j=1}^{n} f_{i,j}(\mathbf{w})$$

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- Previous algorithms: $\mathcal{O}(\log(1/\epsilon))$ communication rounds as long as $\lambda \geq \Omega(1/\sqrt{n})$
- Next:
  - Much simpler approach for randomly partitioned data
  - Can get $\mathcal{O}(\log(1/\epsilon))$ communication rounds as long as $\lambda \geq \tilde{\Omega}(1/n)$
- Detour : without-replacement sampling for stochastic gradient methods...

Stochastic Gradient Methods

$$\min_{\mathbf{w}\in\mathcal{W}\subseteq\mathbb{R}^d} F(\mathbf{w}) = \frac{1}{N}\sum_{i=1}^{N} f_i(\mathbf{w})$$

Stochastic Gradient Descent / Subgradient Method

- $\mathbf{w}_1 = \mathbf{0}$
- For $t = 1, 2, \ldots$
  - Sample i_t
  - Let $\mathbf{g}_t \in \partial f_{i_t}(\mathbf{w}_t)$
  - $\mathbf{w}_{t+1} = \Pi_{\mathcal{W}}\left(\mathbf{w}_t - \eta_t \mathbf{g}_t\right)$

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- Standard analysis: each i_t sampled uniformly from $\{1, \ldots, N\}$
- Works because $\mathbb{E}[\mathbf{g}_t] \in \partial F(\mathbf{w}_t)$
- But: one often samples i_t without replacement
- Equivalently, go over a random shuffle of the data

Why Without Replacement Sampling?

- Often works better (all data equally processed)
- Requires sequential rather than random data access

However, much harder to analyze

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Why Without Replacement Sampling?

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- Incremental gradient bounds (e.g. Nedić and Bertsekas 2001): Hold for any order, but can be exponentially slower
- (Gürbüzbalaban, Ozdaglar, Parillo 2015): For strongly convex and smooth problems, k passes over N data points:
  - Without-replacement: $\mathcal{O}((N/k)^2)$ error
  - With-replacement: $\mathcal{O}(1/Nk)$

New Results

Without-replacement sampling is not worse than with-replacement sampling (in worst-case sense) under various scenarios, e.g.

- $\mathcal{O}(1/\sqrt{T})$ for convex linear prediction
- $\mathcal{O}(1/\lambda T)$ for λ-strongly-convex and smooth linear prediction
- $\exp\left(-\Omega\left(\frac{T}{N+1/\lambda}\right)\right)$ for least squares on N data points, using no-replacement SVRG algorithm

Analysis uses ideas from adversarial online learning and transductive learning theory

Derivation: Convex Linear Prediction

$$\min_{\mathbf{w} \in \mathcal{W} \subseteq \mathbb{R}^d} F(\mathbf{w}) = \frac{1}{N} \sum_{i=1}^{N} f_i(\mathbf{w})$$

Algorithm:

- Sequentially processes $f_{\sigma(1)}, f_{\sigma(2)}, \ldots, f_{\sigma(T)}$ where σ random permutation on $\{1, \ldots, N\}$
- Produces iterates $\mathbf{w}_1, \ldots, \mathbf{w}_T$

Goal: Prove

$$\mathbb{E}\left[\frac{1}{T} \sum_{t=1}^{T} F(\mathbf{w}_t) - F(\mathbf{w}^*)\right] \;\leq\; \mathcal{O}\left(\frac{1}{\sqrt{T}}\right)$$

where $\mathbf{w}^* = \arg\min_{\mathbf{w} \in \mathcal{W}} F(\mathbf{w})$

Derivation: Convex Linear Prediction

Assumption: Algorithm has bounded regret in adversarial online setting: For *any* sequence of convex Lipschitz $f_1, f_2, \ldots$ and $\mathbf{w} \in \mathcal{W}$

$$\frac{1}{T}\sum_{i=1}^{T} f_i(\mathbf{w}_t) - \frac{1}{T}\sum_{i=1}^{T} f_i(\mathbf{w}) \leq \mathcal{O}\left(\frac{1}{\sqrt{T}}\right)$$

Satisfied for e.g. stochastic gradient descent

Derivation: Convex Linear Prediction (1/2)

$$\mathbb{E}_\sigma\left[\frac{1}{T}\sum_{t=1}^{T} F(\mathbf{w}_t) - F(\mathbf{w}^*)\right]$$

Derivation: Convex Linear Prediction (1/2)

$$\mathbb{E}_\sigma \left[\frac{1}{T} \sum_{t=1}^{T} F(\mathbf{w}_t) - F(\mathbf{w}^*) \right]$$

$$= \mathbb{E}_\sigma \left[\frac{1}{T} \sum_{t=1}^{T} F(\mathbf{w}_t) - f_{\sigma(t)}(\mathbf{w}^*) \right]$$

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$$= \mathbb{E}_\sigma\left[\frac{1}{T}\sum_{t=1}^{T}\left(f_{\sigma(t)}(\mathbf{w}_t) - f_{\sigma(t)}(\mathbf{w}^*)\right)\right] + \mathbb{E}_\sigma\left[\frac{1}{T}\sum_{t=1}^{T}\left(F(\mathbf{w}_t) - f_{\sigma(t)}(\mathbf{w}_t)\right)\right]$$

Derivation: Convex Linear Prediction (1/2)

$$
\begin{aligned}
&\mathbb{E}_\sigma\left[\frac{1}{T}\sum_{t=1}^{T} F(\mathbf{w}_t) - F(\mathbf{w}^*)\right] \\
&= \mathbb{E}_\sigma\left[\frac{1}{T}\sum_{t=1}^{T} F(\mathbf{w}_t) - f_{\sigma(t)}(\mathbf{w}^*)\right] \\
&= \mathbb{E}_\sigma\left[\frac{1}{T}\sum_{t=1}^{T} \left(f_{\sigma(t)}(\mathbf{w}_t) - f_{\sigma(t)}(\mathbf{w}^*)\right)\right] + \mathbb{E}_\sigma\left[\frac{1}{T}\sum_{t=1}^{T} \left(F(\mathbf{w}_t) - f_{\sigma(t)}(\mathbf{w}_t)\right)\right] \\
&= \mathcal{O}\left(\frac{1}{\sqrt{T}}\right) + \mathbb{E}_\sigma\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\sum_{i=1}^{N} f_{\sigma(i)}(\mathbf{w}_t)}{N} - \frac{\sum_{i=t}^{N} f_{\sigma(i)}(\mathbf{w}_t)}{N-t+1}\right)\right]
\end{aligned}
$$

Derivation: Convex Linear Prediction (2/2)

Following algebraic manipulations,

$$= \mathcal{O}\left(\frac{1}{\sqrt{T}}\right) + \frac{1}{T}\sum_{t=1}^{T}\frac{t-1}{N}\cdot\mathbb{E}\left[F_{1:t-1}(\mathbf{w}_t) - F_{t:N}(\mathbf{w}_t)\right]$$

where $F_{a:b}$ is average over $f_{\sigma(a)},\ldots,f_{\sigma(b)}$

Derivation: Convex Linear Prediction (2/2)

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$$\leq \mathcal{O}\left(\frac{1}{\sqrt{T}}\right) + \frac{1}{T}\sum_{t=2}^{T}\frac{t-1}{N}\cdot\mathbb{E}\left[\sup_{\mathbf{w}\in\mathcal{W}}\left(F_{1:t-1}(\mathbf{w}) - F_{t:N}(\mathbf{w})\right)\right]$$

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$$\leq \mathcal{O}\left(\frac{1}{\sqrt{T}}\right) + \frac{1}{T}\sum_{t=2}^{T}\frac{t-1}{N}\cdot\mathbb{E}\left[\sup_{\mathbf{w}\in\mathcal{W}}\left(F_{1:t-1}(\mathbf{w}) - F_{t:N}(\mathbf{w})\right)\right]$$

$$\leq \mathcal{O}\left(\frac{1}{\sqrt{T}}\right) + \frac{1}{T}\sum_{t=2}^{T}\frac{t-1}{N}\cdot\mathcal{R}_{t-1:N-t+1}(\mathcal{W}) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right),$$

where $\mathcal{R}_{t-1:N-t+1}(\mathcal{W})$ is transductive Rademacher complexity of $\mathcal{W}$ w.r.t. $f_1, f_2, \ldots, f_N$

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where $\mathcal{R}_{t-1:N-t+1}(\mathcal{W})$ is transductive Rademacher complexity of $\mathcal{W}$ w.r.t. $f_1, f_2, \ldots, f_N$

Specializing to linear predictors and convex Lipschitz losses, get familiar $\mathcal{O}(1/\sqrt{T})$ rate

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$$= \mathcal{O}\left(\frac{1}{\sqrt{T}}\right) + \frac{1}{T}\sum_{t=1}^{T}\frac{t-1}{N}\cdot\mathbb{E}\left[F_{1:t-1}(\mathbf{w}_t) - F_{t:N}(\mathbf{w}_t)\right]$$

where $F_{a:b}$ is average over $f_{\sigma(a)}, \ldots, f_{\sigma(b)}$

$$\leq \mathcal{O}\left(\frac{1}{\sqrt{T}}\right) + \frac{1}{T}\sum_{t=2}^{T}\frac{t-1}{N}\cdot\mathbb{E}\left[\sup_{\mathbf{w}\in\mathcal{W}}\left(F_{1:t-1}(\mathbf{w}) - F_{t:N}(\mathbf{w})\right)\right]$$

$$\leq \mathcal{O}\left(\frac{1}{\sqrt{T}}\right) + \frac{1}{T}\sum_{t=2}^{T}\frac{t-1}{N}\cdot\mathcal{R}_{t-1:N-t+1}(\mathcal{W}) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right),$$

where $\mathcal{R}_{t-1:N-t+1}(\mathcal{W})$ is transductive Rademacher complexity of $\mathcal{W}$ w.r.t. $f_1, f_2, \ldots, f_N$

Specializing to linear predictors and convex Lipschitz losses, get familiar $\mathcal{O}(1/\sqrt{T})$ rate

With more effort, get $\mathcal{O}(1/\lambda T)$ with λ-strong convexity

SVRG

$$\min_{\mathbf{w}\in\mathcal{W}\subseteq\mathbb{R}^d} F(\mathbf{w}) = \frac{1}{N}\sum_{i=1}^{N} f_i(\mathbf{w})$$

- In recent years, fast stochastic algorithms to solve finite-sum problems
  - SAG [Le Roux et al. 2012], SDCA (as analyzed in [Shalev-Shwartz et al. 2013, 2016]), SVRG [Johnson and Zhang 2013], SAGA [Defazio et al. 2014], Finito [Defazio et al. 2014], S2GD [Konecný and Richtárik 2013]...
- Cheap stochastic iterations + linear convergence rate (assuming strong convexity and smoothness)
- All existing analyses based on with-replacement sampling
- We consider without-replacement SVRG

With-Replacement SVRG

Initialize $\tilde{\mathbf{w}}_1 = \mathbf{0}$
for $s = 1, 2, \ldots, S$ **do**
  Compute $\nabla F(\tilde{\mathbf{w}}_s) = \frac{1}{N} \sum_{i=1}^{N} \nabla f_i(\tilde{\mathbf{w}}_s)$
  $\mathbf{w}_1 := \tilde{\mathbf{w}}_s$
  for $t = 1, 2, \ldots, T$ **do**
    Draw i_t uniformly at random from $\{1, \ldots, N\}$
    $\mathbf{w}_{t+1} := \mathbf{w}_t - \eta \left(\nabla f_{i_t}(\mathbf{w}_t) - \nabla f_{i_t}(\tilde{\mathbf{w}}_s) + \nabla F(\tilde{\mathbf{w}}_s) \right)$
  end for
  Pick $\tilde{\mathbf{w}}_{s+1}$ at random from $\mathbf{w}_1, \ldots, \mathbf{w}_T$
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Analysis for 1-smooth, λ-strongly convex functions:
$S = \mathcal{O}(\log(1/\epsilon))$, $T \geq 1/\lambda$

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- Observation [Lee, Lin, Ma 2015]: Can simulate it in distributed setting with randomly partitioned data, as long as $T \leq \mathcal{O}(1/\sqrt{N})$

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- Observation [Lee, Lin, Ma 2015]: Can simulate it in distributed setting with randomly partitioned data, as long as $T \leq \mathcal{O}(1/\sqrt{N})$
- By analysis, only applicable when $\lambda \geq \Omega(1/\sqrt{N})$

Without-Replacement SVRG

Given: Permutation σ on $\{1,\ldots,N\}$
Initialize $\tilde{\mathbf{w}}_1 = \mathbf{0}$
for $s = 1,2,\ldots,S$ **do**
  Compute $\nabla F(\tilde{\mathbf{w}}_s) \;=\; \frac{1}{N}\sum_{i=1}^{N} \nabla f_i(\tilde{\mathbf{w}}_s)$
  $\mathbf{w}_1 \;:=\; \tilde{\mathbf{w}}_s$
  for $t = 1,2,\ldots,T$ **do**
    $\mathbf{w}_{t+1} \;:=\; \mathbf{w}_t - \eta\left(\nabla f_{\sigma(t)}(\mathbf{w}_t) - \nabla f_{\sigma(t)}(\tilde{\mathbf{w}}_s) + \nabla F(\tilde{\mathbf{w}}_s)\right)$
  end for
  Pick $\tilde{\mathbf{w}}_{s+1}$ at random from $\mathbf{w}_1,\ldots,\mathbf{w}_T$
end for

Observation: Can simulate it in distributed setting with randomly partitioned data, all the way up to $T = \Omega(N)$

Bound for Regularized Least Squares

$f_i(\mathbf{w}) = \frac{1}{2}(\langle \mathbf{w}, \mathbf{x}_i \rangle - y_i)^2 + \frac{\hat{\lambda}}{2}\|\mathbf{w}\|^2$; $F(\cdot)$ λ-strongly convex

Theorem

If perform $S = \mathcal{O}(\log(1/\epsilon))$ epochs with $T = \Theta(1/\lambda)$ without-replacement stochastic iterations,

$$\mathbb{E}[F(\tilde{\mathbf{w}}_{S+1}) - F(\mathbf{w}^*)] \leq \epsilon$$

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Implications:

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Implications:

- Non-distributed optimization: $\lambda \geq \tilde{\Omega}(1/N) \Rightarrow \epsilon$-optimal solution without data reshuffling

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Implications:

- Non-distributed optimization: $\lambda \geq \tilde{\Omega}(1/N) \Rightarrow \epsilon$-optimal solution without data reshuffling
- Distributed optimization: $\lambda \geq \tilde{\Omega}(1/n) \Rightarrow \epsilon$-optimal solution with randomly-partitioned data
  - $\mathcal{O}(\log(1/\epsilon))$ communication rounds
  - Runtime dominated by fully-parallelizable gradient computation

Summary and Open Questions

$$\min_{\mathbf{w}\in\mathcal{W}\subseteq\mathbb{R}^d} F(\mathbf{w}) = \frac{1}{n}\sum_{i=1}^{n} F_i(\mathbf{w}) \qquad F_i(\mathbf{w}) = \frac{1}{n}\sum_{j=1}^{n} f_{i,j}(\mathbf{w})$$

- Arbitrary partition ✓
  - Baseline is also (worst-case) optimal...
- δ-related
  - For quadratics ✓
  - More generally X✓
    - Self-concordant losses
    - Algorithms communication-efficient, not necessarily runtime-efficient
- Random partition
  - Least squares ✓
  - More generally X✓
    - λ-strongly-convex and smooth, but only when $\lambda \geq \Omega(1/\sqrt{\text{data size}})$