Quantum Optimal Transport and Sobolev Spaces

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Kinetic and quantum theories

- Classical setting: Phase space $z = (x, v) \in \mathbb{R}^{2d}$
  - Probability distribution $f = f(t, z)$
  - Position density $\varrho_f(x) = \int_{\mathbb{R}^d} f(x, v)\, \mathrm{d}v$
  - pair interaction $K(x - y)$, mean-field potential

$$V_f(x) = (K * \varrho_f)(x) = \int_{\mathbb{R}^d} K(x - y)\, \varrho_f(y)\, \mathrm{d}y$$

  - Vlasov equation

$$\partial_t f + v \cdot \nabla_x f - \nabla V_f \cdot \nabla_v f = 0$$

- Quantum setting: phase space $(x, \boldsymbol{p}) = (x, -i\hbar\nabla)$, with $\hbar = h/(2\pi)$
  - density operator $\boldsymbol{\rho} \in \mathcal{L}^\infty(L^2(\mathbb{R}^d))$

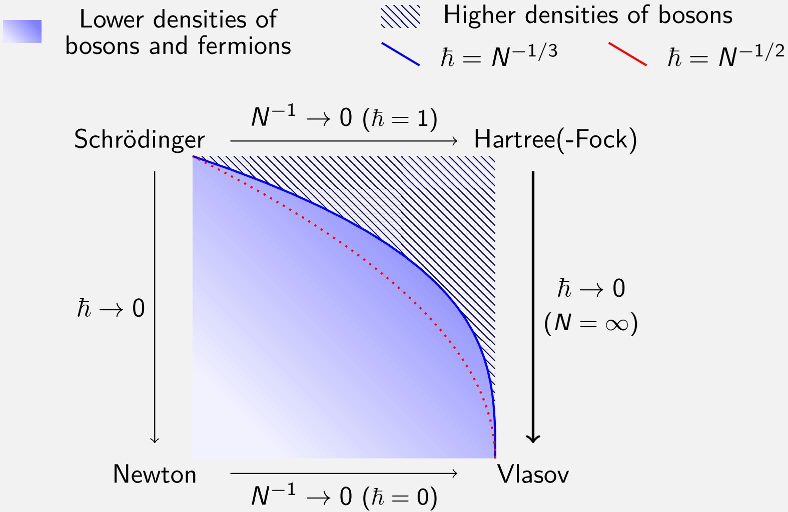
$$\boldsymbol{\rho} \geq 0, \qquad h^d\, \mathrm{Tr}(\boldsymbol{\rho}) = 1$$

  - Position density $\varrho_{\boldsymbol{\rho}}(x) = h^d \boldsymbol{\rho}(x, x)$
  - Hartree(–Fock) equation

$$i\hbar\, \partial_t \boldsymbol{\rho} = [H, \boldsymbol{\rho}] \qquad \text{with } H = \frac{-\hbar^2 \Delta}{2} + V_{\boldsymbol{\rho}} - \mathsf{X}_{\boldsymbol{\rho}}$$

with mean-field potential $V_{\boldsymbol{\rho}} = K * \varrho_{\boldsymbol{\rho}}$

Mean-field and semiclassical limit



Weyl quantization and Wigner transform

- Symmetric quantization of operators

$$\boldsymbol{\rho}_f := \int_{\mathbb{R}^{2d}} \widehat{f}(y,v)\, e^{2i\pi(y\cdot x + w\cdot \boldsymbol{p})} \,\mathrm{d}y\,\mathrm{d}v$$

$$f_{\boldsymbol{\rho}}(x,v) := \int_{\mathbb{R}^{d}} e^{-i\,y\cdot v/\hbar}\, \boldsymbol{\rho}(x+\tfrac{y}{2}, x-\tfrac{y}{2})\,\mathrm{d}y$$

- Analogue of the classical kinetic density (see e.g. [Lions–Paul '93])

$$\partial_t f_{\boldsymbol{\rho}} + v\cdot\nabla_x f_{\boldsymbol{\rho}} = B_h * f_{\boldsymbol{\rho}} \underset{h\to 0}{\longrightarrow} \nabla V_f\cdot\nabla_v f$$

$$h^d\,\mathrm{Tr}((A(x)+B(\boldsymbol{p}))\,\boldsymbol{\rho}) = \iint_{\mathbb{R}^{2d}} (A(x)+B(v))\, f_{\boldsymbol{\rho}}(x,v)\,\mathrm{d}x\,\mathrm{d}v$$

- But
  - not necessarily positive, Lebesgue norms $L^p_{x,v}$ not preserved ($p\neq 2$)
  - Husimi transform $\widetilde{f_{\boldsymbol{\rho}}} = \frac{1}{(\pi\hbar)^d}\, e^{-|z|^2/\hbar} * f_{\boldsymbol{\rho}}$ positive but worse equation

Classical to quantum dictionary

Classical	**Quantum**
$\partial_t f = \{H_f, f\}, H_f = \frac{\lvert v\rvert^2}{2} + V_f$	$i\hbar\,\partial_t \boldsymbol{\rho} = [H_{\boldsymbol{\rho}}, \boldsymbol{\rho}]$
$\int_{\mathbb{R}^{2d}} f\,\lvert v\rvert^n \,\mathrm{d}x\,\mathrm{d}v$	$h^d\,\mathrm{Tr}\big(\lvert \boldsymbol{p}\rvert^n\,\boldsymbol{\rho}\big)$
$\lVert f\rVert_{L^p(\mathbb{R}^{2d})}$	$\lVert\boldsymbol{\rho}\rVert_{\mathcal{L}^p} = h^{\frac{d}{p}}\,\mathrm{Tr}\big(\lvert\boldsymbol{\rho}\rvert^p\big)^{\frac{1}{p}}$
$f(z - z_0)$	$\mathsf{T}_{z_0}\boldsymbol{\rho} = e^{i(\xi_0\cdot x - x_0\cdot\boldsymbol{p})/\hbar}\,\boldsymbol{\rho}\,e^{i(x_0\cdot\boldsymbol{p} - \xi_0\cdot x)/\hbar}$
$(f * g)(z') = \int_{\mathbb{R}^{2d}} f(z)\,g(z'-z)\,\mathrm{d}z$	$f \star \boldsymbol{\rho} = \int_{\mathbb{R}^{2d}} f(z)\,\mathsf{T}_z\boldsymbol{\rho}\,\mathrm{d}z$
$\nabla_x f = \{-v, f\},\ \nabla_v f = \{x, f\}$	$\boldsymbol{\nabla}_x\boldsymbol{\rho} = [\nabla, \boldsymbol{\rho}],\ \boldsymbol{\nabla}_\xi\boldsymbol{\rho} = \left[\frac{x}{i\hbar}, \boldsymbol{\rho}\right]$

$\Longrightarrow$ Quantum Sobolev norms

$$\lVert\boldsymbol{\rho}\rVert_{\dot{\mathcal{W}}^{1,p}} = \lVert\boldsymbol{\nabla}\boldsymbol{\rho}\rVert_{\mathcal{L}^p} \simeq \lVert\boldsymbol{\nabla}_x\boldsymbol{\rho}\rVert_{\mathcal{L}^{1,p}} + \lVert\boldsymbol{\nabla}_\xi\boldsymbol{\rho}\rVert_{\mathcal{L}^{1,p}}$$

Quantum Sobolev inequalities

Gagliardo and Bessel semi-norms

$$\|\boldsymbol{\rho}\|^p_{\dot{\mathcal{W}}^{s,p}} := \gamma_{s,p} \int_{\mathbb{R}^{2d}} \frac{h^d \operatorname{Tr}\left(\left|\mathsf{T}_z\boldsymbol{\rho} - \boldsymbol{\rho}\right|^p\right)}{|z|^{2d+sp}} \,\mathrm{d}z \qquad s \in (0,1)$$

$$\|\boldsymbol{\rho}\|_{\dot{\mathcal{H}}^{s,p}} := \left\|(-\boldsymbol{\Delta})^{s/2}\boldsymbol{\rho}\right\|_{\mathcal{L}^p} \qquad s \in [0,2]$$

with $(-\boldsymbol{\Delta})^s\boldsymbol{\rho} := \boldsymbol{\rho}_{(-\Delta_z)^s f_\rho}$, that is $\boldsymbol{\Delta} = \boldsymbol{\Delta}_x + \boldsymbol{\Delta}_\xi = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$ and

$$(-\boldsymbol{\Delta})^s\boldsymbol{\rho} = c_s \int_{\mathbb{R}^d} \frac{\boldsymbol{\rho} - \mathsf{T}_z\boldsymbol{\rho}}{|z|^{2d+2s}} \,\mathrm{d}z \qquad s \in (0,1)$$

Theorem (LL '22)

Let $s \in [0,1]$, $1 \leq p \leq q < \infty$ such that $\frac{1}{p} - \frac{1}{q} = \frac{s}{2d}$, then there exists $\mathcal{C}_{s,p}$ and $\mathcal{C}'_{s,p}$ independent of $\hbar$ such that for any compact operator $\boldsymbol{\rho}$,

$$\|\boldsymbol{\rho}\|_{\mathcal{L}^q} \leq \mathcal{C}_{s,p} \|\boldsymbol{\rho}\|_{\dot{\mathcal{W}}^{s,p}}$$

$$\|\boldsymbol{\rho}\|_{\mathcal{L}^q} \leq \mathcal{C}'_{s,p} \|\boldsymbol{\rho}\|_{\dot{\mathcal{H}}^{s,p}} \qquad \text{if } p > 1$$

Uncertainty for the skew information

In the case $s = 1$,

$$\|\boldsymbol{\rho}\|_{\mathcal{L}^q} \leq C_{d,p} \, \|\nabla_x \boldsymbol{\rho}\|_{\mathcal{L}^p}^{1/2} \, \|\nabla_\xi \boldsymbol{\rho}\|_{\mathcal{L}^p}^{1/2} \qquad \text{if } \frac{1}{q} = \frac{1}{p} - \frac{1}{2d}$$

Wigner–Yanase skew information: for an operator $A \geq 0$,

$$I_A(\boldsymbol{\rho}) = \frac{1}{2} \operatorname{Tr}\Big(\big|[A, \sqrt{\boldsymbol{\rho}}]\big|^2\Big)$$

Smaller than the variance $I_A(\boldsymbol{\rho}) \leq \sigma_A(\boldsymbol{\rho})^2 := \operatorname{Tr}\Big(\boldsymbol{\rho} \, \big|A - \operatorname{Tr}(\boldsymbol{\rho} A)\big|^2\Big)$, but Sobolev inequality with $p = 2$ still implies

$$\sqrt{I_x(\boldsymbol{\rho}) \, I_{\boldsymbol{p}}(\boldsymbol{\rho})} \geq \frac{1}{8\pi \, \mathcal{C}_{1,2}^2} \, \hbar \, \|\boldsymbol{\rho}\|_{\frac{d}{d-1}}$$

In the case of projection operators $\boldsymbol{\rho} = \boldsymbol{\rho}^2$ in dimension $d = 3$ with $\operatorname{Tr}(\boldsymbol{\rho}) = N$

$$\|[x, \boldsymbol{\rho}]\|_2 \, \|[\nabla, \boldsymbol{\rho}]\|_2 \geq \frac{N^{2/3}}{4\pi \left(\tilde{C}_{1,2}^{\mathsf{S}}\right)^2}$$

Semiclassical convolution

$$f \star \boldsymbol{\rho} = \int_{\mathbb{R}^{2d}} f(z)\, \mathrm{T}_z \boldsymbol{\rho} \,\mathrm{d}z$$

- Convolution inequalities.
  - Young's inequality (Werner '84)
  - Hardy–Littlewood–Sobolev's inequality. Let $(p,q,r) \in (1,\infty)^3$. There exists C independent of $\hbar$ such that for any $f \in L^{q,\infty}$ and $\boldsymbol{\rho} \in \mathcal{L}^r$,

$$\left\| f \star \boldsymbol{\rho} \right\|_{\mathcal{L}^p} \leq C \left\| f \right\|_{L^{q,\infty}} \left\| \boldsymbol{\rho} \right\|_{\mathcal{L}^r} \qquad \text{if } 1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$$

- Toeplitz operators (Wick quantization)

$$\widetilde{\boldsymbol{\rho}}_f = \tfrac{1}{h^d} \int_{\mathbb{R}^{2d}} f(z) \left|\psi_z\right\rangle \left\langle\psi_z\right| \mathrm{d}z = g_h \star \boldsymbol{\rho}_f, \qquad g_h(z) = \tfrac{1}{(\pi\hbar)^d}\, e^{-|z|^2/\hbar}$$

$$\left\| \widetilde{\boldsymbol{\rho}}_f \right\|_{\mathcal{L}^p} \leq \left\| f \right\|_{L^p(\mathbb{R}^{2d})}$$

- *Sobolev inequalities - idea of the proof:* Since $\widetilde{\widetilde{\boldsymbol{\rho}}} = \widetilde{\boldsymbol{\rho}}_{\widetilde{f}_{\boldsymbol{\rho}}}$,

$$\left\| \boldsymbol{\rho} \right\|_{\mathcal{L}^q} \leq \left\| \widetilde{f}_{\boldsymbol{\rho}} \right\|_{L^q(\mathbb{R}^{2d})} + \left\| \boldsymbol{\rho} - \widetilde{\widetilde{\boldsymbol{\rho}}} \right\|_{\mathcal{L}^q}$$

Quantum optimal transport

- Quantum pseudo-distance [Golse–Mouhot–Paul '16]
  - Quantum–quantum coupling $\gamma \in \mathcal{C}(\boldsymbol{\rho}, \boldsymbol{\rho}_2) \subset \mathcal{L}^1(L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d))$

$$h^d \operatorname{Tr}_1(\gamma) = \boldsymbol{\rho}_2, \qquad h^d \operatorname{Tr}_2(\gamma) = \boldsymbol{\rho}$$

  - Analogue to Monge–Kantorovich–Wasserstein distance

$$W_{2,\hbar}(\boldsymbol{\rho}, \boldsymbol{\rho}_2) = \min_{\gamma \in \mathcal{C}(\boldsymbol{\rho}, \boldsymbol{\rho}_2)} h^{2d} \operatorname{Tr}(\boldsymbol{c}_h \gamma) \geq 2\, d\, \hbar$$

with cost $\boldsymbol{c}_h \varphi(y_1, y_2) = \left(|y_1 - y_2|^2 + |\boldsymbol{p}_1 - \boldsymbol{p}_2|^2\right) \varphi(y_1, y_2)$

- Semiclassical pseudo-distance [Golse–Paul '17]
  - Semiclassical coupling $\gamma \in \mathcal{C}(f, \boldsymbol{\rho})$

$$h^d \operatorname{Tr}(\gamma(z)) = f(z), \qquad \int_{\mathbb{R}^{2d}} \gamma(z) \, \mathrm{d}z = \boldsymbol{\rho}$$

  - Classical to quantum pseudo-distance

$$W_{2,\hbar}(f, \boldsymbol{\rho})^2 = \min_{\gamma \in \mathcal{C}(f, \boldsymbol{\rho})} h^d \int_{\mathbb{R}^{2d}} \operatorname{Tr}(\boldsymbol{c}_h(z)\, \gamma(z)) \, \mathrm{d}z \geq d\, \hbar$$

with $\boldsymbol{c}_h(z) \varphi(y) = \left(|x - y|^2 + |v - \boldsymbol{p}|^2\right) \varphi(y)$

Application to mean-field and semiclassical limits

- Regular potentials: ∇K Lipschitz
  - If $f \in L^1(|x|^2 + |v|^2)$ is solution of Vlasov and $\boldsymbol{\rho}$ of Hartree [Golse–Paul '17]

$$W_{2,\hbar}(f, \boldsymbol{\rho}) \leq W_{2,\hbar}(f^{\mathrm{in}}, \boldsymbol{\rho}^{\mathrm{in}})\, e^{\lambda\, t} + C(t)\sqrt{\hbar}$$

  - Mean-field limit: similar statements [Golse–Mouhot–Paul '16, Golse–Paul '17]

- Singular potentials: $K(x) = \frac{\pm 1}{|x|^a}$, with $a \in (-1, 1]$ [LL '19 '21]
  - If f solution of Vlasov with $(1 + |v|^n) f^{\mathrm{in}} \in L^1 \cap L^\infty$ and $\boldsymbol{\rho}$ solution of Hartree with $(1 + |\boldsymbol{p}|^n)\boldsymbol{\rho}^{\mathrm{in}} \in \mathcal{L}^1 \cap \mathcal{L}^\infty$, there exists $T > 0$ such that

$$W_{2,\hbar}(f, \boldsymbol{\rho}) \leq W_{2,\hbar}(f^{\mathrm{in}}, \boldsymbol{\rho}^{\mathrm{in}})^{c(t)}\, e^{\lambda(t)} \qquad \text{for all } t \in [0, T]$$

  - Also works for Hartree–Fock by [Chong–Lafleche–Saffirio '22]

- *Semiclassical limit - idea of the proof*
  - Propagation of moments and weighted norms
  - Interpolation inequality $\|\varrho_{\boldsymbol{\rho}}\|_{L^p} \leq \|\boldsymbol{\rho}\|_{\mathcal{L}^r}^{\theta} \left(h^d\, \mathrm{Tr}(|\boldsymbol{p}|^n\, \boldsymbol{\rho})\right)^{1-\theta}$
  - Adapt [Loeper '06] uniqueness estimate

Quantum optimal pseudometrics properties

Theorem (Golse–Mouhot–Paul '16, Golse–Paul '17 '22)

- *Bound by below*

$$W_2(f, \widetilde{f}_{\boldsymbol{\rho}})^2 \leq W_{2,\hbar}(f, \boldsymbol{\rho})^2 + d\,\hbar$$
$$W_2(\widetilde{f}_{\boldsymbol{\rho}_1}, \widetilde{f}_{\boldsymbol{\rho}_2})^2 \leq W_{2,\hbar}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)^2 + 2\,d\,\hbar$$

- *Special case: Toeplitz operators* $\boldsymbol{\rho} = \widetilde{\boldsymbol{\rho}}_f$

$$W_{2,\hbar}(f, \widetilde{\boldsymbol{\rho}}_g)^2 \leq W_2(f, g)^2 + d\,\hbar$$
$$W_{2,\hbar}(\widetilde{\boldsymbol{\rho}}_f, \widetilde{\boldsymbol{\rho}}_g)^2 \leq W_2(f, g)^2 + 2\,d\,\hbar$$

- *Triangle inequalities*

$$W_{2,\hbar}(f, \boldsymbol{\rho}) \leq W_2(f, g) + W_{2,\hbar}(g, \boldsymbol{\rho})$$
$$W_{2,\hbar}(\boldsymbol{\rho}, \boldsymbol{\rho}_2) \leq W_{2,\hbar}(\boldsymbol{\rho}, g) + W_{2,\hbar}(g, \boldsymbol{\rho}_2)$$

In particular,

$$W_{2,\hbar}(f, \widetilde{\boldsymbol{\rho}}_f)^2 = d\,\hbar \quad \text{and} \quad W_{2,\hbar}(\widetilde{\boldsymbol{\rho}}_f, \widetilde{\boldsymbol{\rho}}_f)^2 = 2\,d\,\hbar$$

Quantum optimal transport versus Sobolev norms

Theorem (LL '23)

- *"Self-distance bound"*

$$W_{2,\hbar}(\widetilde{f}_{\boldsymbol{\rho}}, \boldsymbol{\rho})^2 \leq d\,\hbar + \hbar^2 \left\|\boldsymbol{\nabla}\sqrt{\boldsymbol{\rho}}\right\|_{\mathcal{L}^2}^2$$

$$W_{2,\hbar}(\boldsymbol{\rho}, \boldsymbol{\rho})^2 \leq 4\,d\,\hbar + 4\,\hbar^2 \left\|\boldsymbol{\nabla}\sqrt{\boldsymbol{\rho}}\right\|_{\mathcal{L}^2}^2$$

- *If* $0 \leq \boldsymbol{\rho}, \boldsymbol{\rho}_2 \leq 1$ *then*

$$\left\|\boldsymbol{\rho} - \boldsymbol{\rho}_2\right\|_{\dot{\mathcal{H}}^{-1}} < W_{2,\hbar}(\boldsymbol{\rho}, \boldsymbol{\rho}_2) + 2\sqrt{d\,\hbar}$$

- *If* $h^d \operatorname{Tr}\big((|x|^n + |\boldsymbol{p}|^n)\,(\boldsymbol{\rho} + \boldsymbol{\rho}_2)\big)$ *is bounded uniformly in* $\hbar$

$$W_{2,\hbar}(\boldsymbol{\rho}, \boldsymbol{\rho}_2) \leq C\left\|\boldsymbol{\rho} - \boldsymbol{\rho}_2\right\|_{\dot{\mathcal{W}}^{-1,1}}^{\theta} + \sqrt{2\,d\,\hbar} + \hbar\left\|\boldsymbol{\nabla}\sqrt{\boldsymbol{\rho}}\right\|_{\mathcal{L}^2} + \hbar\left\|\boldsymbol{\nabla}\sqrt{\boldsymbol{\rho}_2}\right\|_{\mathcal{L}^2}$$

$\Longrightarrow$ Small self-distances for thermal states, powers of Toeplitz operators, Slater determinants ...

- *Self-distance bound - Idea of proof:* Use the coupling

$$\gamma(z) := \boldsymbol{\rho}^{1/2}\,\mathsf{T}_z\boldsymbol{\rho}_{g_h}\,\boldsymbol{\rho}^{1/2}$$

The case of projection operators

- Slater determinants
  - $\boldsymbol{\rho}_N = |\Psi_N\rangle\langle\Psi_N|$ with $Nh^d = 1$ and $\Psi_N(x_1, \dots x_N) := \frac{1}{\sqrt{N!}} \det(\psi_j(x_k))_{1\leq j,k\leq N}$
  - First marginal is not "semiclassically smooth" [LL '23]

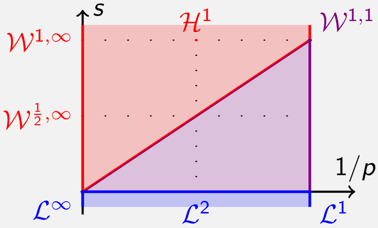
$$\boldsymbol{\rho}_{N:1} = \sum_{j=1}^{N} |\psi_j\rangle\langle\psi_j| \quad \Longrightarrow \quad \boldsymbol{\rho}_{N:1}^2 = \boldsymbol{\rho}_{N:1} \text{ and } \boldsymbol{\rho}_{N:1} \notin \mathcal{W}^{s,p}, \quad s > \tfrac{1}{p}, \; p > 1$$

  - Spectral functions with V nice [Fournais–Mikkelsen '20]

$$\boldsymbol{\rho}_V := \mathbb{1}_{\mathbb{R}_-}(-\hbar^2\Delta + V) \in \mathcal{W}^{1,1} \implies \|\boldsymbol{\nabla}\boldsymbol{\rho}_V\|_{\mathcal{L}^2} \leq C_V/\sqrt{\hbar}$$

  - By the "self-distance bound" for $W_{2,\hbar}$,

$$W_{2,\hbar}(\widetilde{f}_{\boldsymbol{\rho}}, \boldsymbol{\rho})^2 \leq d\,\hbar + C_V\,\hbar$$



Thank you for your attention!

Schatten norm estimates

Theorem (LL, Saffirio '20)

If f is initially sufficiently smooth, $\boldsymbol{\rho}$ a solution of Hartree(–Fock) equation and $\boldsymbol{\rho}_f$ is the Weyl quantization of f, then

$$\|\boldsymbol{\rho} - \boldsymbol{\rho}_f\|_{\mathcal{L}^1} \leq \left(\left\|\boldsymbol{\rho}^{\mathrm{in}} - \boldsymbol{\rho}_f^{\mathrm{in}}\right\|_{\mathcal{L}^1} + C_f(t)\,\hbar\right) e^{\lambda_f(t)}$$

Classical case: weak-strong uniqueness

$$\|f_1 - f_2\|_{L^1(\mathbb{R}^6)} \leq \left\|f_1^{\mathrm{in}} - f_2^{\mathrm{in}}\right\|_{L^1(\mathbb{R}^6)} \exp\left(C\int_0^T \|\nabla_v f_2\|_{L^{3,1}_x L^1_v}\,\mathrm{d}t\right),$$

Proof: $(\partial_t + v\cdot\nabla_x - \nabla V_1\cdot\nabla_v)(f_1 - f_2) = (\nabla V_1 - \nabla V_2)\cdot\nabla_v f_2$, so that, since $V = K * \varrho$, we obtain

$$\partial_t \int_{\mathbb{R}^{2d}} |f_1 - f_2|\,\mathrm{d}x\,\mathrm{d}v = -\int_{\mathbb{R}^{2d}} (\varrho_1 - \varrho_2)\;\nabla K * \int_{\mathbb{R}^d} \operatorname{sgn}(f_1 - f_2)\,\nabla_v f_2\,\mathrm{d}v\,\mathrm{d}x$$

$$\leq \|f_1 - f_2\|_{L^1}\left\|\nabla K * \int_{\mathbb{R}^d} |\nabla_v f_2|\,\mathrm{d}v\right\|_{L^\infty},$$

Mean-field limit

Theorem (Chong–LL–Saffirio '21)

Let ρ be a solution of the Hartree–Fock equation initially smooth in a semiclassical sense. Then there exists $k, T > 0$, $\rho^{\mathrm{in}}_{N,\rho} \in \mathcal{L}^1(\mathcal{F})$ such that for any ρ_N solution of Schrödinger equation with initial condition $\rho^{\mathrm{in}}_N \in \mathcal{L}^1(\mathcal{F})$ commuting with $\mathcal{N}$, for any $t \in [0, T]$

$$\|\rho_{N:1} - \rho\|_{\mathcal{L}^1} \lesssim \frac{C\, e^{\lambda\, t}}{N^{1/2}} \left(1 + \left\|(\mathcal{N} + N)^k \left(\rho^{\mathrm{in}}_N - \rho^{\mathrm{in}}_{N,\rho}\right)\right\|_{\mathcal{L}^1(\mathcal{F})}\right)$$

where λ is independent of $\hbar$ if $a < 1/2$.

- Other $\mathcal{L}^p$ estimates possible with $p > 1$ and other rates
- Schrödinger to Vlasov?
  - For $a < 1/2$ and "regular" mixed states
    Example: thermal states [Chong–LL–Saffirio '21]
  - Pure states cannot be that regular!

Main steps of the proof of the mean-field

- Uniform-in-$\hbar$ regularity for the Hartree–Fock equation
  - Let $m = 1 + |\boldsymbol{p}|^n$ with $n \geq 3$ and $\boldsymbol{\rho}$ such that

$$\boldsymbol{\rho}^{\text{in}}, \sqrt{\boldsymbol{\rho}^{\text{in}}} \in \mathcal{L}^\infty(m) \cap \mathcal{W}^{1,2}(m) \cap \mathcal{W}^{1,4}(m).$$

    Then there exists $T > 0$ such that uniformly in $\hbar$ it holds

$$\boldsymbol{\rho}, \sqrt{\boldsymbol{\rho}} \in L^\infty([0,T], \mathcal{L}^\infty(m) \cap \mathcal{W}^{1,2}(m) \cap \mathcal{W}^{1,4}(m)),$$

- Mean-Field
  - Purification of mixed states: take an appropriate square root

$$\boldsymbol{\rho}_N \in \mathcal{L}^1(\mathcal{F}(L^2)) \to \boldsymbol{\nu}_N \in \mathcal{L}^2(\mathcal{F}(L^2)) \text{ such that } |\boldsymbol{\nu}_N|^2 = \boldsymbol{\rho}_N$$

  - Identify the kernel of the operator with a function of a double Fock space

$$\boldsymbol{\nu}_N \in \mathcal{L}^2(\mathcal{F}(L^2)) \to \Psi_{\boldsymbol{\rho}_N} \in \mathcal{F}\left(L^2 \oplus L^2\right)$$

  - Bogoliubov transformation $\mathrm{R}_t = \mathrm{R}_{\boldsymbol{\rho}_t}$

$$\Psi_t = \mathrm{R}_t^* \, e^{i\,t\,\mathrm{L}_N} \, \mathrm{R}_t \Psi^{\text{in}}$$

  - Number of particles outside the Bogoliubov state $\boldsymbol{\rho}_{N,\boldsymbol{\rho}}$ such that

$$\Psi_{\boldsymbol{\rho}_{N,\boldsymbol{\rho}}} = \mathrm{R}_t^* \, e^{it\mathrm{L}_N} \, \mathrm{R}_t \Omega$$

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