A new hyperbolic dispersive model for coastal waves

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From Navier-Stokes to depth-averaged equations

Consider Navier-Stokes/ Euler equations of incompressible fluid

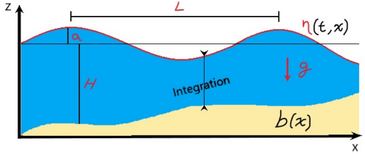
$$\begin{cases} \textbf{div}\,(\boldsymbol{u}) = 0, & \text{(Continuity Eq)} \\ \dfrac{\partial \boldsymbol{u}}{\partial t} + \textbf{div}\,(\boldsymbol{u} \otimes \boldsymbol{u}) = \boldsymbol{g} - \dfrac{1}{\rho}\textbf{grad}\; p + \nu \boldsymbol{u}, & \text{(Momentum Eq)} \end{cases}$$

where ρ: density, ν: kinematic viscosity, p: pressure, and $\boldsymbol{g}$: gravitational acceleration.

Numerical resolution is time-consuming $\Rightarrow$ Integrating the equations over the depth

Depth-averaged water waves models

$$\varepsilon = H/L \text{ (shallowness)}$$
$$\mu = a/H \text{ (nonlinearity)}$$



BC. no-slip, kinematic, dynamic condition

$$\begin{cases} \dfrac{\partial h}{\partial t} + \textbf{div}\,(h\boldsymbol{U}) = 0, & \text{(Mass Eq)} \\ \dfrac{\partial h\boldsymbol{U}}{\partial t} + \textbf{div}\left(h\boldsymbol{U} \otimes \boldsymbol{U} + \dfrac{gh^2}{2}\mathcal{I} + P_{NH}\,\mathcal{I}\right) = 0, & \text{(Momentum Eq)} \end{cases}$$

model	**NSWE** $\mathcal{O}(\varepsilon)$	$\mathcal{O}(\varepsilon\mu)$	**SGN** $\mathcal{O}(\varepsilon^2)$
Pressure	$P_{NH} = 0$	Boussinesq	$P_{NH} = h^2\ddot{h}/3$
μ	no assump		no assump
Type	hyperbolic		dispersive

Depth-averaged water waves models

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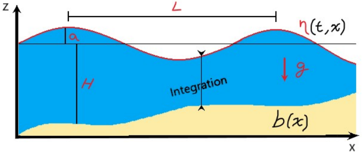

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Gavrilyuk&Favrie, **2017**; Escalante, **2019,2020**; Richard, **2021**

Breaking wave: Energy dissipation

Energy dissipation due to breaking wave should be included in depth-averaged context.

Classic methods are:

1. Extra terms in the mass and/or the momentum equation to provide a necessary dissipation.

   Svendsen, **1984, 1996**; Zelt, **1991**; Schäffer et al., **1993**; Musumeci et al., **2005** etc.

2. Switching or hybrid methods (natural dissipation through the shock)

   Bonneton et al., **2011**; Tissier et al., **2012**; Kazolea et al., **2014**; Duran & Marche, **2015,2017**

Both methods need a **breaking criterion** to determine when the wave breaks.

New hyperbolic model for breaking waves

Objectives:

1. New hyperbolic model with cheaper numerical cost
2. Model which is capable to capture breaking phenomenon
3. Model with good dispersive properties

Assumptions

Mimic large-eddy simulation (LES), decompose the velocity

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + \overbrace{\text{small-scale turbulence}}^{\text{modeled by turbulent viscosity}}$$

SW eq. SGN eq.

$$\overline{u}(t,x,z) = U(t,x) + u'(t,x,z)$$

$$p = p_{Hydro} + p_{Nonhydro}$$

Model of Kazakova & Richard

- u' – the vertical variation of the horizontal velocity is arbitrary
- Hypothesis of weakly turbulent flow from Teshukov (2007): $u' \approx O(\varepsilon)$
- The non-hydrostatic pressure is taken into account
- No surface tension and shear stress at the free surface

Breaking waves model

Model derivation: equations for h,U,W

Dimensionless equations $\xrightarrow{\int}$ $\xrightarrow{O(\varepsilon^2)}$ + dissipation

Residual stress tensor is modeled by turbulent viscosity ν_T and shear stress

Mass equation:

$$\frac{\partial \tilde{h}}{\partial \tilde{t}} + \frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{x}} = 0$$

Ox-Momentum equation:

$$\frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}}\left(\tilde{h}\tilde{U}^2 + \frac{\tilde{h}^2}{2} + \varepsilon^2 \tilde{h}\left\langle \tilde{u}'^2 \right\rangle + \varepsilon^2 \tilde{h}\left\langle \tilde{p}_N \right\rangle - \varepsilon^2 \boxed{2\tilde{\nu}_T \tilde{h}\frac{\partial \tilde{U}}{\partial \tilde{x}}}\right) = -\tilde{p}(b)\frac{\partial \tilde{b}}{\partial \tilde{x}}$$

Oz-Momentum equation:

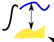
$$\frac{\partial \tilde{h}\tilde{W}}{\partial \tilde{t}} + \frac{\partial \tilde{h}\tilde{U}\tilde{W}}{\partial \tilde{x}} = \tilde{p}_N(b)$$

Defining $\tilde{\varphi}$ (enstrophy), $\tilde{W}$, and $\tilde{P}$

$$\tilde{\varphi} := \frac{\left\langle \tilde{u}'^2 \right\rangle}{\tilde{h}^2} \equiv \frac{1}{\tilde{h}^3}\int_{\tilde{b}}^{\tilde{\eta}} \tilde{u}'^2 d\tilde{z}, \quad \overbrace{\tilde{W} = \langle \tilde{w} \rangle, \ \tilde{P} = \langle \tilde{p}_N \rangle}^{\text{hyperbolic structure}}$$

Breaking waves model

Model derivation: equations for P and φ

Energy equation  $\xrightarrow{O(\varepsilon^2)}$

Hyperbolicity admits slight compressibility in non-hydrostatic pressure!

$$\frac{h^2}{2}\left(\frac{\partial h\varphi}{\partial t}+\frac{\partial hU\varphi}{\partial x}\right)+\frac{\partial h\langle e_a\rangle}{\partial t}+\frac{\partial hU\langle e_a\rangle}{\partial x}$$

$$=-h\left\langle P^r\right\rangle-\left(hP-2\nu_T h\frac{\partial U}{\partial x}\right)\frac{\partial U}{\partial x}-2\left(P+2\nu_T\frac{\partial U}{\partial x}\right)(W-\dot{b}),$$

where e_a is the acoustic energy and P^r is the energy transfer from large-scale turbulence to small-scale turbulence.

Postulate $\langle e_a\rangle$ + decouple $\longrightarrow$ equations for P and φ

$$\frac{1}{h}\int_b^\eta e_a dz=\langle e_a\rangle=\frac{P^2}{2a_c^2}$$

where a_c is the constant sound velocity.

Breaking waves model

Full system of equations

Under the mild slope condition

$$\frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0$$

$$\frac{\partial hU}{\partial t} + \frac{\partial}{\partial x}\left(hU^2 + \frac{gh^2}{2} + h^3\varphi + hP\right) = \frac{\partial}{\partial x}\left(2\nu_T h\frac{\partial U}{\partial x}\right) - gh\frac{\partial b}{\partial x}$$

$$\frac{\partial hW}{\partial t} + \frac{\partial hUW}{\partial x} = \frac{3}{2}P + 3\nu_T\frac{\partial U}{\partial x}$$

$$\frac{\partial hP}{\partial t} + \frac{\partial hUP}{\partial x} = -a_c^2\left(h\frac{\partial U}{\partial x} + 2W\right)$$

$$\frac{\partial h\varphi}{\partial t} + \frac{\partial hU\varphi}{\partial x} = -\frac{2}{h}\langle P^r\rangle + \frac{4\nu_T}{h}\left(\frac{\partial U}{\partial x}\right)^2 - \frac{8\nu_T W}{h^2}\frac{\partial U}{\partial x}$$

Breaking waves model

Full system of equations

Without turbulence and enstrophy φ

$$\frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0$$

$$\frac{\partial hU}{\partial t} + \frac{\partial}{\partial x}\left(hU^2 + \frac{gh^2}{2} + hP\right) = -gh\frac{\partial b}{\partial x}$$

$$\frac{\partial hW}{\partial t} + \frac{\partial hUW}{\partial x} = \frac{3}{2}P$$

$$\frac{\partial hP}{\partial t} + \frac{\partial hUP}{\partial x} = -a_c^2\left(h\frac{\partial U}{\partial x} + 2W\right)$$

Hyperbolic model of the SGN equation developed by 📖 Richard, **2021** in an incompressible context.

Breaking waves model

Full system of equations

Acoustic sound velocity $a_c \to \infty$

$$\frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0$$

$$\frac{\partial hU}{\partial t} + \frac{\partial}{\partial x}\left(hU^2 + \frac{gh^2}{2} + h^3\varphi + \frac{h^2\ddot{h}}{3}\right) = \frac{\partial}{\partial x}\left(4\nu_T h\frac{\partial U}{\partial x}\right) - gh\frac{\partial b}{\partial x}$$

$$\frac{\partial h\varphi}{\partial t} + \frac{\partial hU\varphi}{\partial x} = -\frac{2}{h}\langle P^r\rangle + \frac{8\nu_T}{h}\left(\frac{\partial U}{\partial x}\right)^2$$

Breaking wave model developed by 📖 Kazakova & Richard, **2019**.

Further without turbulence and enstrophy would reduce to SGN equation

$$\frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0$$

$$\frac{\partial hU}{\partial t} + \frac{\partial}{\partial x}\left(hU^2 + \frac{gh^2}{2} + \frac{h^2\ddot{h}}{3}\right) = -gh\frac{\partial b}{\partial x}$$

Hyperbolicity

The primitive form of the system has the Jacobian matrix

$$\begin{pmatrix} U & h & 0 & 0 & 0 \\ g + 3h\varphi + \dfrac{P}{h} & U & 0 & 1 & h^2 \\ 0 & 0 & U & 0 & 0 \\ 0 & a^2 & 0 & U & 0 \\ 0 & 0 & 0 & 0 & U \end{pmatrix}$$

The eigenvalues are

$$\lambda = U(\text{triple roots}),\ U \pm \sqrt{gh + 3h^2\varphi + P + a^2},$$

with the corresponding eigenvectors form a basis in $\mathbb{R}^5$. The model is hyperbolic if $h > 0$.

Asymptotic dispersion relation to SGN model

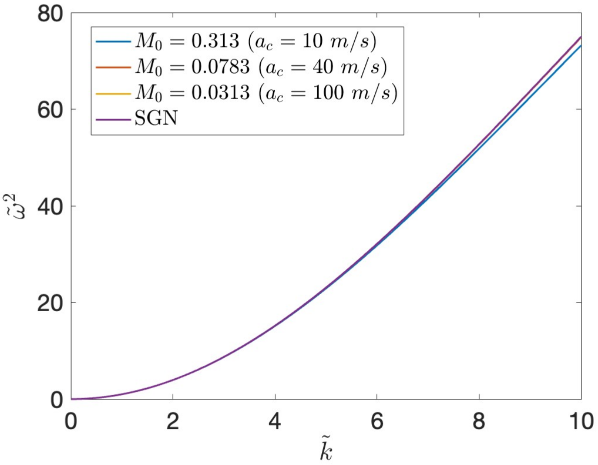
The dispersion relation of the model is

$$\frac{M_0^2}{3}\tilde{\omega}^4 - \tilde{\omega}^2\left[1 + \frac{\tilde{k}^2}{3}\left(1 + M_0^2\right)\right] + \tilde{k}^2 = 0$$

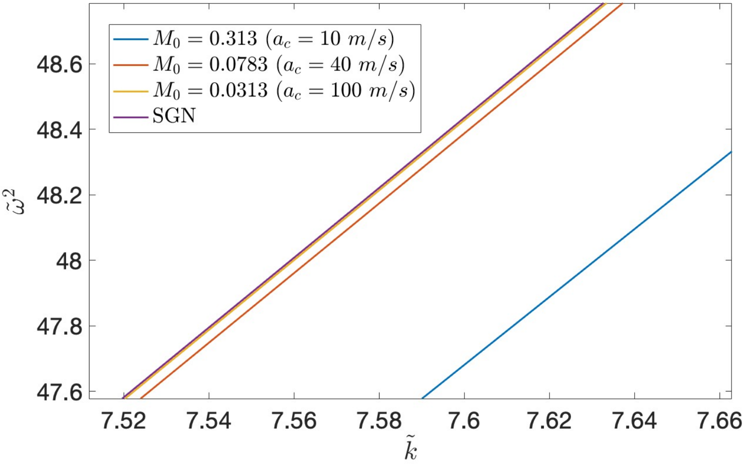
where $M_0 = \sqrt{gh_0}/a_c$ is the Mach number at the reference state. As $a_c \to \infty$, the dispersion relation of the model approaches to that of SGN model

$$\tilde{\omega}^2 = \frac{\tilde{k}^2}{1 + \frac{\varepsilon^2\tilde{k}^2}{3}}$$

Asymptotic dispersion relation to SGN model



Asymptotic dispersion relation to SGN model



From energy dissipation to breaking criterion

Recall

$$\frac{\partial hU}{\partial t} + \frac{\partial}{\partial x}\left(hU^2 + \frac{gh^2}{2} + h^3\varphi + hP\right) = \frac{\partial}{\partial x}\left(2\nu_T h\frac{\partial U}{\partial x}\right) - gh\frac{\partial b}{\partial x}$$

$$\frac{\partial hW}{\partial t} + \frac{\partial hUW}{\partial x} = \frac{3}{2}P + 3\nu_T\frac{\partial U}{\partial x}$$

$$\frac{\partial h\varphi}{\partial t} + \frac{\partial hU\varphi}{\partial x} = -\frac{2}{h}\langle P^r\rangle + \frac{4\nu_T}{h}\left(\frac{\partial U}{\partial x}\right)^2 - \frac{8\nu_T W}{h^2}\frac{\partial U}{\partial x}$$

Mean of dissipation $\langle P^r\rangle$ and the turbulent viscosity ν_T have the forms

$$\langle P^r\rangle = \frac{C_r}{2}h^2\varphi^{3/2},\ \nu_T = \frac{h^2\sqrt{\varphi}}{R},$$

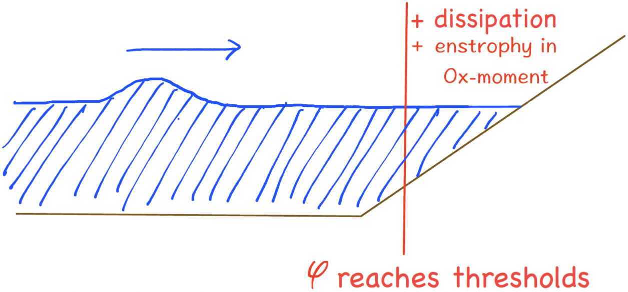
where C_r is a dimensionless quantity and R is an analogue of Reynold's number. (See Kazakova & Richard, **2019**)

Breaking criterion explain

If there's NO criterion, we resolve the full system from the very beginning. **The dissipation is too significant for some cases.** Therefore, a breaking criterion is needed.

The enstrophy φ has definition similar to vorticity magnitude. Good for capturing the turbulence generated by the breaking.

Numerical treatment of breaking:



Breaking criteria

Kazakova&Richard, **2019**

$$\varphi_0 = \frac{g}{h_0}\widetilde{\varphi}_0, \quad \widetilde{\varphi}_0 = \begin{cases} \left(0.1 + \frac{0.031}{\mu_0}\right), & \mu_0 > 0.05 \\ 0, & \mu_0 < 0.05 \end{cases}$$

Criterion: Activate once $\tilde{\varphi} > \tilde{\varphi}_0$.

New criterion depends on the local dimensionless quantities

$$\tilde{\varphi}_1 = \frac{\varphi\eta}{g} > \alpha_1, \qquad \tilde{\varphi}_2 = \frac{\varphi h}{g} > \alpha_2,$$

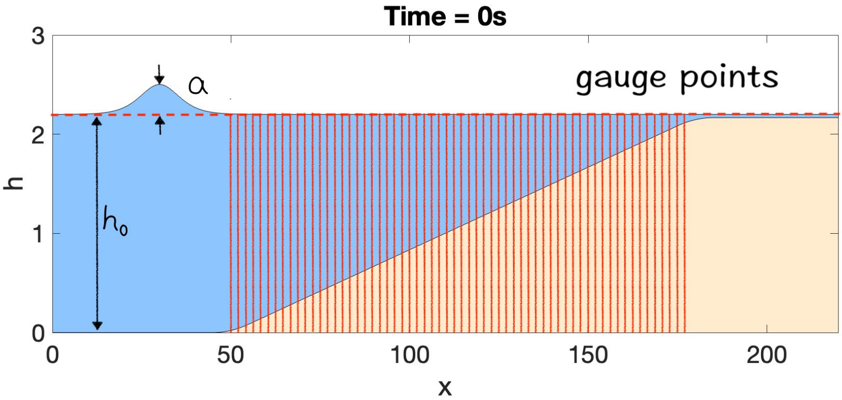
where $\alpha_1 > \alpha_2$ are two thresholds for the activation of the breaking.

Criterion: Activate once $\tilde{\varphi}_1 > \alpha_1$ and deactivate when $\tilde{\varphi}_2 > \alpha_2$.

Note. the new criterion now depends only on local variables/quantaties!

Experiment of Hsiao *et. al*, 2008

Gauge points of the Experiment 📖 Hsiao *et. al*, **2008**



Numerical Comparison: Hsiao Trial 41

Free surface and breaking criteria

In this scenario, we take trial 41 for example. The wave conditions are

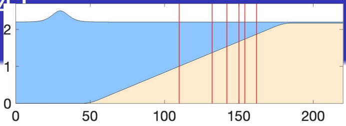
$$h_0 = 2.2\ m,\ \ \mu = 0.137$$

The numerical parameters we take

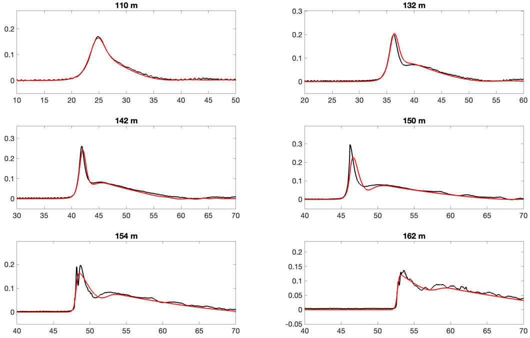
$$R = 1.7,\ \ \alpha_1 = 0.09,\ \ \alpha_2 = 0.005$$

Numerical Comparison: Hsiao Trial 41

Gauge points comparison



Amplitude



Breaking position:

real:148m

numerical:148.7m

Time series

Numerical Comparison

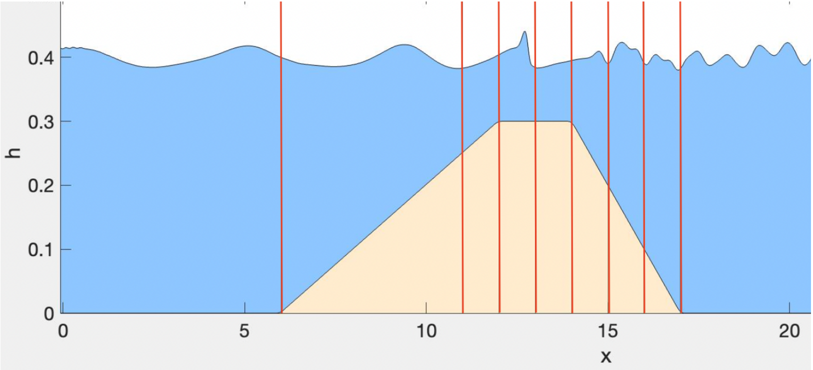
Deviation in position and amplitude of breaking point

Trial	Breaking position	Breaking amplitude
3	$-0.3/-0.7$	$-0.0496/-0.0423$
9	$0.9/1$	$-0.0341/-0.0345$
14	$-0.4/0$	$-0.101/-0.0983$
15	$0.9/1.4$	$-0.0899/-0.0866$
17	$1.2/1.7$	$-0.112/-0.1083$
19	$1.2/1.7$	$-0.0978/-0.0942$
21	$2.4/1$	$0.0206/0.018$
25	$0.6/0.3$	$0.006/0.002$
31	$0.3/0.1$	$-0.022/-0.027$
37	$0.4/0.4$	$-0.0815/-0.082$
41	$0.7/0.9$	$-0.1018/-0.1008$
43	$1.1/1.4$	$-0.076/-0.074$
49	$0.4/-0.6$	$-0.057/-0.057$
54	$0.6/0.3$	$-0.101/-0.1064$
Average	$0.7143/0.6357$	$-0.0641/-0.0637$

Blue data is with the criteria of Kazakova & Richard that depends on initial global μ_0. Red data is with new local breaking criteria.

Experiments of Beji & Battjes, 1993

Experimental setup of 📖 Beji & Battjes, **1993**



Numerical Comparison: Beji & Battjes SLS case

Free Surface and breaking criteria

In this scenario, we take Sinusoidal Long Spilling (SLS) wave for example. The wave conditions are

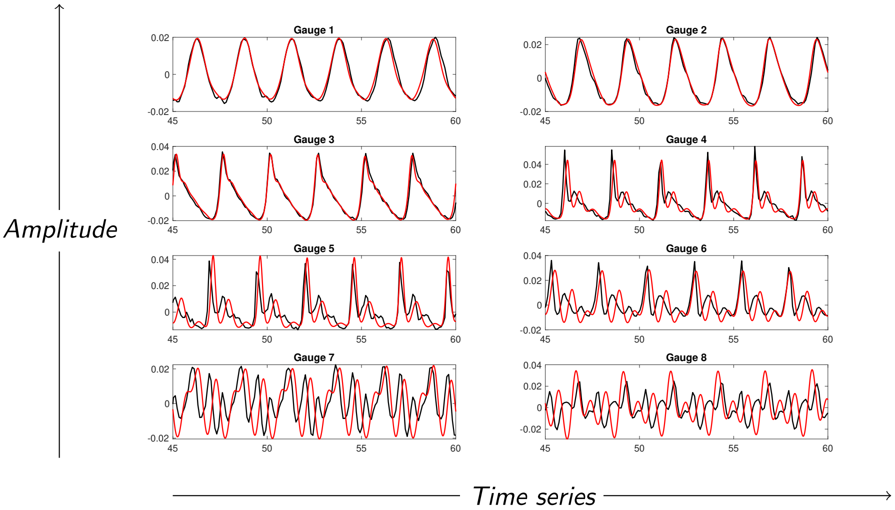
$$T = 2.5\ s,\ a = 0.016\ m$$

The numerical parameters we take

$$R = 7.5,\ \alpha_1 = 0.01,\ \alpha_2 = 0.005$$

Numerical Comparison: Beji & Battjes SLS case

Gauge points comparison



Model with improved dispersive properties

Follow 📖 Bonneton et al., **2011**, the idea is to use w at some height above the bottom as a variable instead of the average vertical velocity

$$\checkmark \quad w|_{z=b+\frac{\alpha}{2}h} \qquad \times \ \langle w \rangle$$

The choice of α follows 📖 Bonneton et al., **2011** with an optimal value

$$\alpha = 1.159$$

Improved dispersion relation

Airy wave theory gives

$$\tilde{\omega}^2 = \tilde{k}\tanh(\tilde{k})$$

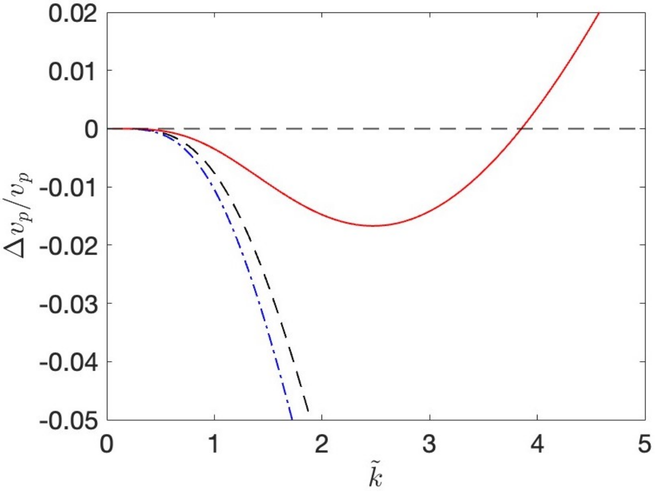
accurate for shallow water and deep water.

The dispersion relation of the model with improved dispersive properties is

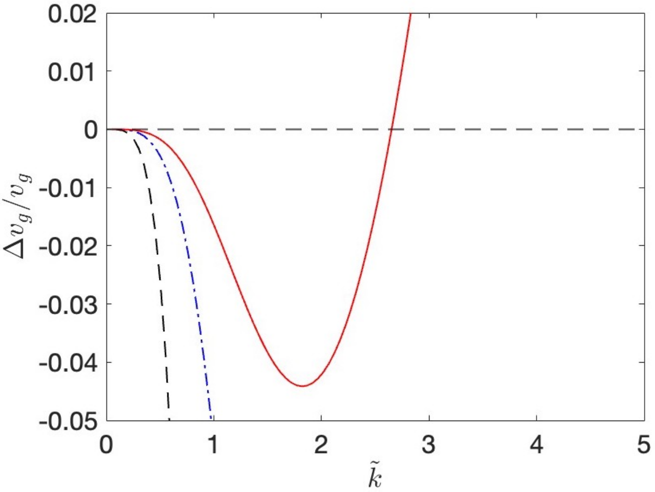
$$\frac{M_0^2}{3}\tilde{\omega}^4 - \tilde{\omega}^2\left[1 + \frac{\tilde{k}^2}{3}\left(\alpha + \frac{2\alpha - 1}{\alpha}M_0^2\right)\right] + \tilde{k}^2\left[1 + \tilde{k}^2\frac{\alpha - 1}{3}\left(1 + \frac{M_0^2}{\alpha}\right)\right] = 0$$

where $M_0 = \sqrt{gh_0}/a_c$ is the Mach number at the reference state.

Improved dispersion relation

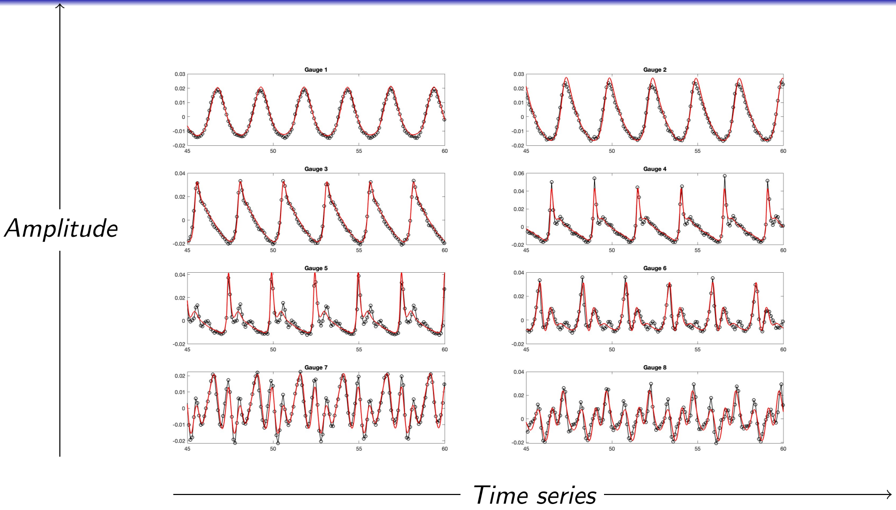


Improved dispersion relation



Numerical Comparison: Beji & Battjes SLS case

Gauge points comparison



Numerical Comparison: Beji & Battjes Irregular wave

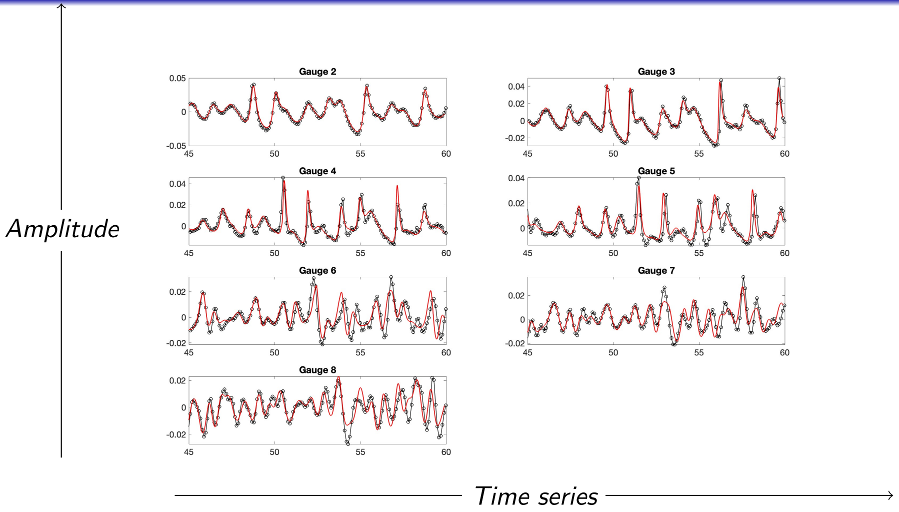
Free Surface and breaking criteria

In this scenario, irregular waves are generated by JONSWAP stectrum, the breaker is of spilling type. The numerical parameters we take

$$R = 7.5,\ \alpha_1 = 0.01,\ \alpha_2 = 0.005$$

Numerical Comparison: Beji & Battjes Irregular wave

Gauge points comparison



Conclusions & Perspectives

Conclusions:

- Capable to capture breaking phenomenon
- Improved dispersive property
- Hyperbolic structure gives cheaper numerical cost
- Validated by the comparison to several experiments
- Local breaking criterion

Perspectives:

- More stable numerical scheme can be implemented (A. Duran, in prep.)
- Sediment transport coupling (Julien Chauchat, LEGI)
- Implementation in TOLOSA (https://tolosa-project.com)

Merci de votre attention