Prediction-correction pedestrian flow via minimum flow problem

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Joint work with N.Igbida and G.Jradi

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Univ.Grenoble, Grenoble INP*, LJK

Plan of the talk

- Overview and some remainders.
- Presentation of the model.
- Numerical illustrations.


Sortie

Overview and some remainders

Crowd motion models

Microscopic models

- Each person can be tracked individually.
- Usually based on ODEs.
- Literature: Helbing, Blue and Adler, Kirchner and Schadschneider, Maury and Venel...

Taken from: `http://www.cromosim.fr/`

Crowd motion models

Macroscopic models

- The population is presented as whole via a density ρ.
- Usually based on PDEs, optimal transport.
- Literature: Hughes, Bellomo and Dogbé, Kirchner and Schadschneider, Maury *et al.*, ...

Taken from A.R-Chupins' thesis

Macroscopic models

- Density $0 \leq \rho \leq \rho_{\mathsf{max}}$ (usually = 1).
- Conservation law: $\partial_t \rho + \mathsf{div}(\rho U[\rho]) = 0$.

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Hughes ~ 2000

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$$\|\nabla \Phi\| = 1/f(\rho).$$

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Given velocity V, take $U[\rho] = \mathbf{Proj}_{K(\rho)}(V)$

$$K(\rho) = \left\{ v \in L^2(\Omega)^2, \int_\Omega v\nabla q : \ \forall q \in H^1_+(\Omega), q = 0 \text{ a.e on } [\rho < 1] \right\}.$$

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$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho U[\rho]) = 0 \text{ in } Q := (0,T) \times \Omega \\ U[\rho] = \mathbf{Proj}_{K(\rho)}(V) \\ \rho(0) = \rho_0 \text{ in } \Omega \end{cases}$$

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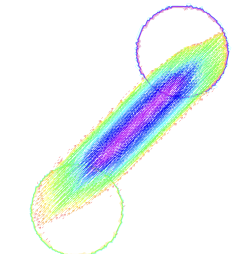
$$\begin{cases} \partial_t \rho - \Delta p + \operatorname{div}(\rho V) = 0 \text{ in } Q \\ \rho(0) = \rho_0 \text{ in } \Omega \\ 0 \leq \rho \leq 1, p \geq 0, p(1-\rho) = 0. \ + \text{BC on } p \end{cases}$$

Presentation of the model

Minimum flow problem

- Introduced by Martin Beckmann in 1952 as a model for transportation.
- Transference in urbain area:

$$(\mathrm{BK}): \quad \inf_{\Phi \in L^1(\Omega)^N} \left\{ \int_\Omega |\Phi(x)| \,\mathrm{d}x : \quad -\operatorname{div}(\Phi) = \mu_1 - \mu_2 \text{ in } \mathcal{D}'(\Omega) \right\}.$$



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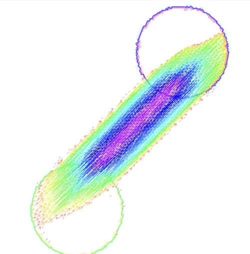

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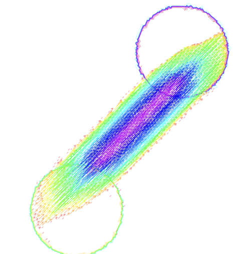
- transportation cost by consumer



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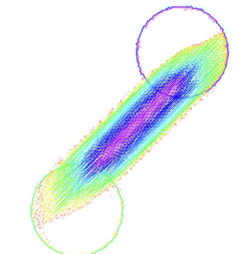
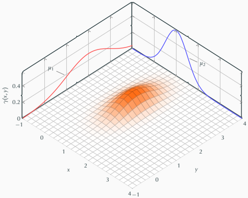
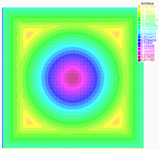
- consumers traffic flow
- transportation cost by consumer
- measure of excess demand



Relation to optimal transport



Transport plan γ



Kantorovich potential u

$$(\text{MK}): \quad \min\left\{ \int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma(x,y) : \gamma \in \Gamma(\mu_1,\mu_2) \right\}.$$

$$(\text{KR}): \quad \sup\left\{ \int_{\Omega} u \,\mathrm{d}(\mu_1-\mu_2) : \ |\nabla u| \leq 1 \right\}.$$

Left to right: G.Monge, L.Kantorovich and M.Beckmann

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$$\inf_{\rho,\Phi} \left\{ \int_\Omega F(x, \Phi(x)) \mathrm{d}x : \ \rho \in \mathcal{K}, \Phi \in L^1(\Omega)^N, \ -\mathrm{div}(\Phi) = \rho_{k+\frac{1}{2}} - \rho \text{ in } \mathcal{D}'(\Omega) \right\}$$

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Examples of cost functions

- Homogenous case: $F(x, \xi) = |\xi|$.
- Quadratic case $F(x, \xi) = \frac{1}{2}|\xi|^2$.
- More generally a Finsler metric.

Duality: case $F(x,\xi) = |\xi|$

Théorème (EIJ 2023)

For any $0 \le \tilde\rho \in L^s(\Omega)$ *, we have*

$$\inf_{(\rho,\Phi)} \left\{ \int_\Omega |\Phi(x)|\, \mathrm{d}x : \ \Phi \in \mathcal{F}(\tilde\rho - \rho) \text{ and } \rho \in \mathcal{K} \right\}$$
$$= \max_{p\in\mathcal{G}} \left\{ \int_\Omega \tilde\rho\, p\, \mathrm{d}x - \int_\Omega p^+\, \mathrm{d}x \right\}$$

where

$$\mathcal{G} := \left\{ z \in W^{1,\infty}(\Omega) \ : \ z = 0 \text{ on } \Gamma_D \text{ and } |\nabla z(x)| \le k \text{ a.e. } x \in \Omega \right\},$$

$$\mathcal{K} = \{\rho \in L^\infty(\Omega) \ : \ 0 \le \rho \le 1 \text{ a.e. in } \Omega\},$$

and

$$\mathcal{F}(\tilde\rho - \rho) := \left\{ \Phi \in L^s(\Omega)^N \ : \ -\mathrm{div}(\Phi) = \tilde\rho - \rho \text{ in } \Omega \right\}.$$

Duality: case $F(x,\xi) = |\xi|$

Idea of the proof

Define on $\mathcal{M}_b(\overline{\Omega})$ the following functional

$$\mathcal{H}(h) = \inf_{\Phi} \left\{ \int_\Omega |\Phi(x)| \mathrm{d}x : \ \mathcal{F}(\tilde{\rho} - \rho + h) \right\}$$

$\rightsquigarrow \mathcal{H}$ is convex and l.s.c.

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$\rightsquigarrow$ $\mathcal{H}$ is convex and l.s.c.

$\rightsquigarrow$ Use that $\mathcal{H}^{**} = \mathcal{H}$, where:

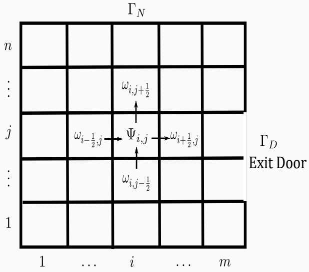
$$\mathcal{H}^*(p) = \sup_q \langle p, q \rangle - \mathcal{H}(q).$$

Numerical approximation and examples

Discretization

- Room $\Omega \subset \mathbb{R}^2, (\partial\Omega = \Gamma_N \cup \Gamma_D)$.
- We define discrete divergence is defined by:

$$(\mathrm{div}_h \, \Phi)_{i,j} = \frac{\Phi^1_{i+\frac{1}{2},j} - \Phi^1_{i-\frac{1}{2},j}}{h} + \frac{\Phi^2_{i,j+\frac{1}{2}} - \Phi^2_{i,j-\frac{1}{2}}}{h}.$$



Discretization

Subdivise $[0, T]$ into subintervals $[t_k, t_{k+\frac{1}{2}}]$ and $[t_{k+\frac{1}{2}}, t_{k+1}]$, with $k = 0, \ldots, n-1$. On each interval $[t_k, t_{k+\frac{1}{2}})$ we solve the continuity equation

$$\begin{cases} \overbrace{\partial_t \rho}^{\text{FD}} + \overbrace{\operatorname{div}(V\rho)}^{FV} = 0 \\ \rho(t_k) = \rho^k, \end{cases}$$

where $V = (V^x, V^y)$ is the velocity field given by $V = -\nabla \mathbf{D}/\|\nabla \mathbf{D}\|$, and $\mathbf{D}$ solves:

$$\begin{cases} \|\nabla \mathbf{D}\| = f & \text{in } \Omega, \\ \mathbf{D} = 0 & \text{on } \Gamma_D. \end{cases}$$

Discretization of the minimum flow problem

On each $[t_{k+\frac{1}{2}}, t_{k+1})$ we solve

$$\inf_{(\rho,\Phi)} \left\{ \int_\Omega |\Phi(x)| \mathrm{d}x \textbf{:} \ -\operatorname{div}(\Phi) = \rho^{k+\frac{1}{2}} - \rho \textbf{ in } \Omega, \ \Phi \cdot \nu = 0 \textbf{ on } \Gamma_N, 0 \leq \rho \leq 1 \right\}.$$

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We rewrite it as

$$(\mathrm{M}): \quad \inf_{(\rho,\Phi)} \mathcal{A}(\rho,\Phi) + \mathcal{B}(\Lambda(\rho,\Phi))$$

where

$$\mathcal{A}(\rho,\Phi) = \int_\Omega |\Phi(x)| \mathrm{d}x + \mathbb{I}_{[0,1]}(\rho), \quad \Lambda(\rho,\Phi) = \rho - \operatorname{div}\Phi \quad \text{and } \mathcal{B} = \mathbb{I}_{\{\rho^{k+\frac{1}{2}}\}}.$$

Discretization of the minimum flow problem

Discrete problem:

$$(\mathrm{M})_d: \quad \min_{(\rho,\Phi)} \mathcal{A}_h(\rho,\Phi) + \mathcal{B}_h(\Lambda_h(\rho,\Phi)),$$

where

$$\mathcal{A}_h(\rho,\Phi) = h^2 \sum_{i=1}^{m+1} \sum_{j=1}^{n+1} \|\Phi_{i,j}\| + \mathbb{I}_{[0,1]}(\rho) \text{ and } \mathcal{B}_h = \mathbb{I}_{\mathcal{C}}.$$

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Here:

$$\mathcal{C} := \left\{ (a_{i,j}) \; a_{i,j} = \rho_{i,j}^{k+\frac{1}{2}}, \;\; \forall 1 \leq i \leq m, \; 1 \leq j \leq n \right\} \text{ and } \Lambda_h(\rho,\Phi) = \rho - \mathrm{div}_h \, \Phi.$$

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Primal-dual form:

$$\min_{(\rho,\Phi)} \max_{p} \mathcal{A}_h(\rho,\Phi) + \langle u, \Lambda_h(\rho,\Phi)\rangle - \mathcal{B}_h^*(p),$$

$\rightsquigarrow$ Chambolle-Pock's primal-dual algorithm.

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We only need to compute the "proximal" operators of $\mathcal{A}_h$ and $\mathcal{B}_h$.

Discretization of the minimum flow problem

Reminder

$$\mathbf{Prox}_E(p) = \operatorname*{argmin}_q \frac{1}{2}\|p-q\|^2 + E(q).$$

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Steps of (PD) algorithm:

- Initialization: Take $\alpha, \beta > 0$, $\rho_0, \Phi^0, p_0, \bar{p}^0$.
- Primal-Dual updates:

$$\begin{aligned}(\rho^{l+1}, \Phi^{l+1}) &= \mathbf{Prox}_{\beta \mathcal{A}_h}\left((\rho^l, \Phi^l) - \beta \Lambda_h^*(\bar{p}^l)\right);\\ p^{l+1} &= \mathbf{Prox}_{\alpha \mathcal{B}_h^*}\left(p^l + \alpha \Lambda_h(\rho^{l+1}, \Phi^{l+1})\right);\end{aligned}$$

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Convergence: $\alpha\beta\|\Lambda_h\|^2 < 1$

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Computing the proximal operators:

- We have $\mathcal{A}_h(\rho,\Phi) = \mathbb{I}_{[0,1]}(\rho) + \|\Phi\|_1$, so,

$$\left(\mathbf{Prox}_{\mathcal{A}_h}(\rho,\Phi)\right)_{i,j} = \left(\max(0,\min(1,\rho_{i,j})), \max(0, 1-\tfrac{1}{|\Phi_{i,j}|})\Phi_{i,j}\right)$$

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- To compute $\mathbf{Prox}_{\alpha\mathcal{B}_h^*}$, we make use of Moreau's identity

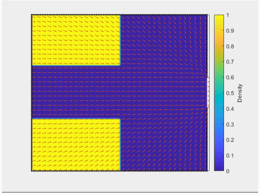
$$p = \mathbf{Prox}_{\alpha\mathcal{B}_h^*}(p) + \alpha\mathbf{Prox}_{\alpha^{-1}\mathcal{B}_h}(p/\alpha).$$

We get,

$$\left(\mathbf{Prox}_{\alpha\mathcal{B}_h^*}(p)\right)_{i,j} = \left(p_{i,j} - \alpha\mathbf{Proj}_{\mathcal{C}_{i,j}}(p_{i,j}/\alpha)\right).$$

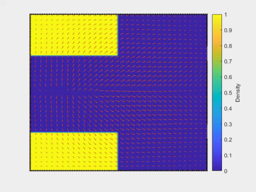
Simulations

The initial density is $\rho_0(\mathbf{x}) = \mathbf{1}_{S_1}(\mathbf{x}) + \mathbf{1}_{S_2}(\mathbf{x})$ with $S_1 = [0, \frac{1}{2}] \times [0, \frac{1}{3}]$, $S_2 = [0, \frac{1}{2}] \times [\frac{2}{3}, 1]$, $\Gamma_D = \{1\} \times [0.4, 0.6]$ and $f(\mathbf{x}) \equiv 1$.



Simulations

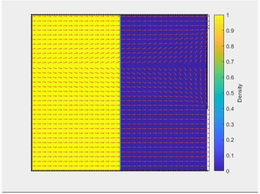
With the same $\rho_0(\mathbf{x})$, $\Gamma_D = (\{1\} \times [0.2, 0.3]) \cup (\{1\} \times [0.7, 0.8])$ and $f(\mathbf{x}) = e^{-3\times\left((x-\frac{1}{2})^2+(y-\frac{1}{2})^2\right)}$.



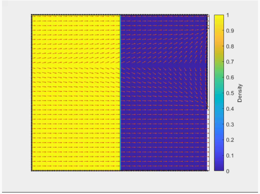
Homogeneous case vs quadratic case

- We compare our model $F(x,\xi) = |\xi|$, with $F(x,\xi) = \frac{1}{2}|\xi|^2$.
- Take $\rho_0(\mathbf{x}) = \mathbf{1}_{S_1}(\mathbf{x})$ with

$$S_1 = [0, \frac{1}{2}] \times [0,1] \text{ and } \Gamma_D = (\{1\} \times [0,0.4]) \cup (\{1\} \times [0.9,1]) \,.$$



(a) $F(x,\xi) = |\xi|$

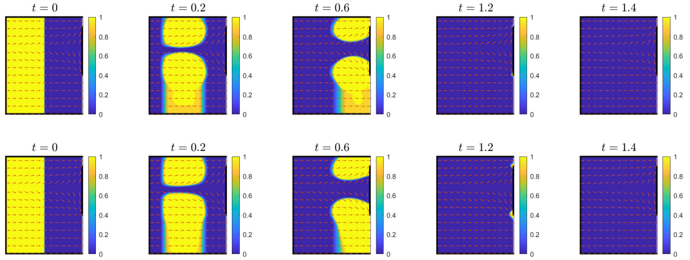


(b) $F(x,\xi) = \frac{1}{2}|\xi|^2$

Homogeneous case vs quadratic case

- We compare our model $F(x,\xi) = |\xi|$, with $F(x,\xi) = \frac{1}{2}|\xi|^2$.
- Take $\rho_0(\mathbf{x}) = \mathbf{1}_{S_1}(\mathbf{x})$ with

$$S_1 = [0, \frac{1}{2}] \times [0,1] \text{ and } \Gamma_D = (\{1\} \times [0,0.4]) \cup (\{1\} \times [0.9,1])\,.$$



(a) Top line: $F(x,\xi) = |\xi|$. Bottom line: $F(x,\xi) = \frac{1}{2}|\xi|^2$

Thank you for your attention!