Invariant Gibbs Measures for the 1d Nonlinear Schrödinger Equation with Trapping Potential

Van Duong Dinh

ENS de Lyon

(joint work with Nicolas Rougerie (ENS de Lyon), Leonardo Tolomeo (University of Edinburgh), and Yuzhao Wang (University of Birmingham))

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Outline

What is Gibbs measure?

▶ **In finite dimensional setting.** Consider an ODE on $\mathbb{R}^{2d}$:

$$\begin{cases} \partial_t x_j = \frac{\partial H}{\partial \xi_j}, \\ \partial_t \xi_j = -\frac{\partial H}{\partial x_j}, \end{cases} \quad j = 1, \dots, d, \tag{ODE}$$

with a Hamiltonian $H(x,\xi) = H(x_1, \dots, x_d, \xi_1, \dots, \xi_d)$.

- The Gibbs measure associated to (ODE) is given by

$$d\mu(x,\xi) = \frac{1}{Z} e^{-H(x,\xi)} dx d\xi,$$

  where Z is the normalized constant.

- μ is **invariant** under the flow of (ODE), i.e., *for A measurable set of* $\mathbb{R}^{2d}$*,* $\mu(A) = \mu(\Phi(t)A)$ *for all* $t \in \mathbb{R}$ because
  - vector field $(\partial_\xi H, -\partial_x H)$ is divergence-free → (Liouville's theorem) Lebesgue measure $dx d\xi$ is invariant under (ODE);
  - conservation of Hamiltonian, i.e., $H(x,\xi)$ is independent of time.

What is Gibbs measure?

▶ **In infinite dimensional setting.** Consider a NLS on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$:

$$i\partial_t u + \partial_x^2 u + u = \pm |u|^{p-2} u, \quad p > 2. \tag{NLS}$$

- (NLS) also has a Hamiltonian structure, i.e., $\partial_t u = -i\frac{\partial H}{\partial \overline{u}}$ with the Hamiltonian

$$H(u) = \frac{1}{2}\int_{\mathbb{T}} |\partial_x u|^2 + |u|^2 \pm \frac{1}{p}\int_{\mathbb{T}} |u|^p.$$

- It is expected that the Gibbs measure of the form

$$d\mu(u) = \frac{1}{Z} e^{-H(u)} du$$

  is invariant under the flow of (NLS).
  ⚠ **There is no infinite dimensional Lebesgue measure.**
- The construction and invariance of Gibbs measures for NLS (and other dispersive PDEs) have been studied by many mathematicians: **Lebowitz, Rose, Speer, Bourgain, Tzvetkov, Burq, Oh, Nahmod, Killip, Visan, Deng, Tolomeo** ,...
- Why do we care about invariant Gibbs measure? **a sort of "conservation law" to obtain global solutions at low regularity.**

1d NLS with trapping potential

- We consider 1d nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u - Vu = \pm |u|^{p-2} u, \quad x \in \mathbb{R} \tag{NLS}$$

with $p > 2$ and the trapping potential $V : \mathbb{R} \to \mathbb{R}_+$ satisfying

$$V(x) \sim |x|^s \quad (s > 0) \quad \text{as} \quad |x| \to \infty.$$

- Our goal is to construct the following, formally defined, Gibbs measures

$$d\mu(u) = \frac{1}{Z} \exp\left(-\frac{1}{2} \int_{\mathbb{R}} |\partial_x u|^2 + V|u|^2 \mp \frac{1}{p} \int_{\mathbb{R}} |u|^p \right) du$$

and to prove their invariance under the flow of (NLS).

Motivation

- Physics: The many-body mean-field approximation of Bose-Einstein condensates:
  - the defocusing (+) measure (**Lewin, Nam, Rougerie '15, '18** ; **Fröhlich, Knowles, Schlein, Sohinger '17, '19**).
  - the focusing (-) measure (**Sohinger, Rout '22, '23**).

- Mathematics: Most work on Gibbs measures relies on the explicit knowledge of eigenfunctions of the linear operator $-\Delta + V(x)$, e.g.
  - torus (plan waves) **Lebowitz, Rose, Speer '88** ; **Bourgain '94** ; **Oh, Quastel '13** ; **Oh, Sosoe, Tolomeo '22** ,...
  - harmonic potential $V(x) = |x|^2$ (link with Hermite polynomials) **Burq, Thomann, Tzvetkov '13** ; **Deng '12** ; **Robert, Seong, Tolomeo, Wang '22** .
  - disk/sphere (link with Bessel functions) **Tzvetkov '06, '08** ; **Bourgain, Bulut '14** .

  For $V(x) \sim |x|^s$ **at infinity**, such an **explicit knowledge on eigenfunctions is not available**.

1d NLS on torus

One can formally think it as (NLS) with $s = \infty$, $x \in [0,1]$ and a periodic boundary condition.

- defocusing: **McKean '95** .
- focusing:

$$d\mu(u) = \frac{1}{Z}\exp\left(-\frac{1}{2}\int_{\mathbb{T}} |\partial_x u|^2 + |u|^2 + \frac{1}{p}\int_{\mathbb{T}} |u|^p\right)\mathbb{1}_{\{\int_{\mathbb{T}} |u|^2 \le m\}} du$$

- **Lebowitz, Rose, Speer '88** proved
  - normalizability ($Z < +\infty$) for $2 < p < 6$ and any $m > 0$;
  - non-normalizability ($Z = +\infty$)
    + for $p > 6$ and any $m > 0$;
    + for $p = 6$ and $m > \|Q\|^2_{L^2(\mathbb{R})}$, where Q is the unique (up to symmetries) optimizer of the Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|^6_{L^6(\mathbb{R})} \le C_{\text{opt}}\|\partial_x u\|^2_{L^2(\mathbb{R})}\|u\|^4_{L^2(\mathbb{R})}.$$

- **Bourgain '94** proved
  - normalizability
    + for $2 < p < 6$ and any $m > 0$;
    + for $p = 6$ and $m > 0$ small;
  - invariance for $p > 2$ and $p \le 6$ for the focusing nonlinearity.
- **Oh, Sosoe, Tolomeo '22** proved the normalizability for $p = 6$ and $0 < m < \|Q\|^2_{L^2(\mathbb{R})}$ and $m = \|Q\|^2_{L^2(\mathbb{R})}$. This is remarkable since NLS admits blowup solutions with the minimal mass $\|Q\|^2_{L^2(\mathbb{R})}$ (**Ogawa, Tsutsumi '90**).

1d NLS with harmonic potential $V(x) = |x|^2$

► **Burq, Thomann, Tzvetkov '13** constructed the Gibbs measures for
- defocusing with $p > 2$;
- focusing

$$d\mu(u) = \frac{1}{Z} \exp\left(-\frac{1}{2} \int_{\mathbb{R}} |\partial_x u|^2 + |x|^2 |u|^2 + \frac{1}{p} \int_{\mathbb{R}} |u|^p \right) \mathbb{1}_{\{\int_{\mathbb{R}} : |u|^2 : \leq m\}} du$$

only with $p = 4$ and any $m > 0$ (the renormalized mass cutoff).
They also proved the invariance for p **even integer**.

► **Robert, Seong, Tolomeo, Wang '22** considered the **focusing** measures and proved
- normalizability for $2 < p < 6$ and any $m > 0$;
- non-normalizability for $p \geq 6$ and any $m > 0$.

⚠ **No critical nonlinearity.**

Main results

Theorem 1 (Construction of Gibbs measures) D., Rougerie '23, D., Rougerie, Tolomeo, Wang '23+

- *Normalizability for* **defocusing** *nonlinearity with* $p > \max\{2, \frac{4}{s}\}$;
- **Focusing** *nonlinearity*
  - *Super-harmonic* $(s > 2)$

$$d\mu(u) = \frac{1}{Z}\exp\left(-\frac{1}{2}\int_{\mathbb{R}}|\partial_x u|^2 + V|u|^2 + \frac{1}{p}\int_{\mathbb{R}}|u|^p\right)\mathbb{1}_{\{\int_{\mathbb{R}}|u|^2\leq m\}}du$$

  - *normalizability*
    - + *for* $2 < p < 6$ *and any* $m > 0$;
    - + *for* $p = 6$ *and* $0 < m < \|Q\|^2_{L^2(\mathbb{R})}$, *where Q is the unique (up to symmetries) optimizer of the Gagliardo-Nirenberg-Sobolev inequality*.
  - *non-normalizability*
    - + *for* $p = 6$ *and* $m > \|Q\|^2_{L^2(\mathbb{R})}$;
    - + *for* $p > 6$ *and any* $m > 0$.
  - *harmonic* $(s = 2)$

$$d\mu(u) = \frac{1}{Z}\exp\left(-\frac{1}{2}\int_{\mathbb{R}}|\partial_x u|^2 + V|u|^2 + \frac{1}{p}\int_{\mathbb{R}}|u|^p\right)\mathbb{1}_{\{\int_{\mathbb{R}}:|u|^2:\leq m\}}du$$

  - *normalizability for* $2 < p < 6$ *and any* $m > 0$.
  - *non-normalizability for* $p \geq 6$ *and any* $m > 0$.

Main results

Theorem 1 (Construction of Gibbs measures, continue)

- *sub-harmonic* ($1 < s < 2$)

$$d\mu(u) = \frac{1}{Z}\exp\left(-\frac{1}{2}\int_{\mathbb{R}}|\partial_x u|^2 + |x|^2|u|^2 + \frac{\alpha}{p}\int_{\mathbb{R}}|u|^p\right)\mathbb{1}_{\{\int_{\mathbb{R}}:|u|^2:\leq m\}}du$$

  - *normalizability*
    - *for* $\frac{4}{s} < p < 2+2s$, *any* $\alpha > 0$ *and any* $m > 0$;
    - *for* $p = 2+2s$, $0 < \alpha < \alpha_0$ *with* **some** $\alpha_0 > 0$ *and any* $m > 0$.
  - *non-normalizability*
    - *for* $p = 2+2s$, $\alpha > \alpha_0$ *and any* $m > 0$;
    - *for* $p > 2+2s$, *any* $\alpha > 0$ *and any* $m > 0$.

Theorem 2 (Invariance of Gibbs measures)

The above Gibbs measures are invariant under the flow of (NLS) provided

- *super-harmonic* ($s > 2$): $2 < p < 4+s$;
- *(sub)-harmonic* ($1 < s \leq 2$): $\frac{4}{s} < p < 6$.

Consequently, (NLS) is globally well-posed almost surely on the support of Gibbs measures.

▶ **Open questions**:

? normalizability for $m = \|Q\|^2_{L^2(\mathbb{R})}$ (for $s > 2$) and $\alpha = \alpha_0$ (for $1 < s < 2$);

? invariance for $p \geq 4+s$ (for $s > 2$) and $p \geq 6$ (for $1 < s \leq 2$).

Sobolev spaces

- Denote
$$H = -\partial_x^2 + V(x).$$
Since V is trapping, the spectral theorem yields
$$H = \sum_{j\geq 1} \lambda_j |u_j\rangle\langle u_j|,$$
where
$$0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_j \to +\infty$$
are eigenvalues of H and $\{u_j\}_{j\geq 1}$ are the corresponding normalized eigenfunctions which form an orthonormal basis of $L^2(\mathbb{R})$.

- Let $1 \leq p \leq \infty$ and $\beta \in \mathbb{R}$. Sobolev spaces associated to H are defined by
$$\mathscr{W}^{\beta,p}(\mathbb{R}) = \left\{u \in \mathscr{S}'(\mathbb{R}) : H^{\beta/2}u \in L^p(\mathbb{R})\right\}.$$
When $p = 2$, we write $\mathscr{W}^{\theta,2}(\mathbb{R}) = \mathscr{H}^{\theta}(\mathbb{R})$.

- **Equivalence norms (Yajima, Zhang '01)** :
For $\beta > 0$ and $1 < p < \infty$,
$$\|H^{\beta/2}u\|_{L^p(\mathbb{R})} \sim \|\langle D\rangle^{\beta}u\|_{L^p(\mathbb{R})} + \|\langle x\rangle^{\beta s/2}u\|_{L^p(\mathbb{R})},$$
where $\langle D\rangle = \sqrt{1-\partial_x^2}$.

Gaussian measure

- By writing

$$d\mu(u) = \frac{1}{Z}\exp\left(\mp\frac{1}{p}\int_{\mathbb{R}}|u|^p\right)\exp\left(-\frac{1}{2}\int_{\mathbb{R}}|\partial_x u|^2 + V|u|^2\right)du,$$

we aim at defining μ as an absolutely continuous probability measure with respect to the **Gaussian measure** formally given by

$$d\rho(u) = \frac{1}{C}\exp\left(-\frac{1}{2}\int_{\mathbb{R}}|\partial_x u|^2 + V|u|^2\right)du.$$

- Observe that

$$\int_{\mathbb{R}}|\partial_x u|^2 + V|u|^2 = \langle u, Hu\rangle_{L^2} = \|u\|^2_{\mathscr{H}^1}.$$

For $u \in \mathscr{S}'(\mathbb{R})$, we decompose

$$u = \sum_{j\geq 1}\alpha_j u_j, \quad \alpha_j = \langle u_j, u\rangle_{L^2} \in \mathbb{C}$$

and write

$$d\rho(u) = \frac{1}{C}\exp\left(-\frac{1}{2}\sum_{j\geq 1}\lambda_j|\alpha_j|^2\right)\prod_{j\geq 1}d\alpha_j.$$

- We can think of defining ρ as

$$d\rho(u) = \prod_{j\geq 1}\frac{\lambda_j}{2\pi}e^{-\frac{1}{2}\lambda_j|\alpha_j|^2}d\alpha_j \tag{GM}$$

with

$$C = \prod_{j\geq 1}\frac{2\pi}{\lambda_j}.$$

Gaussian measure

- To define rigorously the Gaussian measure ρ, we will take the limit $\Lambda \to \infty$ of the finite dimensional Gaussian measure

$$d\rho_\Lambda(u) = \prod_{\lambda_j \le \Lambda} \frac{\lambda_j}{2\pi} e^{-\frac{1}{2}\lambda_j |\alpha_j|^2} d\alpha_j$$

defined on

$$E_\Lambda = \text{span}\{u_j : \lambda_j \le \Lambda\}.$$

- **But, taking the limit on which topology?**

- to view ρ_Λ as an induced probability measure under the randomization

$$u_\Lambda^\omega = \sum_{\lambda_j \le \Lambda} \frac{g_j(\omega)}{\sqrt{\lambda_j}} u_j,$$

where $\{g_j\}_{j\ge1}$ are i.i.d. complex-valued **standard** Gaussian random variables ($\mathcal{N}_\mathbb{C}(0,1)$) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- to show

$$\mathbb{E}_\mathbb{P}[\| u_\Theta^\omega - u_\Lambda^\omega \|^2_{\mathcal{H}^\theta}] = \sum_{\Lambda < \lambda_j \le \Theta} \lambda_j^{\theta-1} \to 0 \text{ as } \Theta, \Lambda \to \infty$$

for a suitable θ. For that, we need

$$\sum_{j\ge1} \lambda_j^{\theta-1} < \infty.$$

Regularity of Gaussian measure

- **Under which condition on θ, we have $\sum_{j\geq 1} \lambda_j^{\theta-1} < \infty$?**
  - For 1d torus, $\lambda_j \sim j^2$, so we need $\theta < \frac{1}{2}$.
  - For 1d harmonic potential, $\lambda_j \sim j$, hence we require $\theta < 0$.
- For general trapping potential, we use the **Lieb–Thirring inequality** (**Dolbeault, Felmer, Loss, Paturel '06**): for $p > \frac{1}{2}$,

$$\mathrm{Tr}[H^{-p}] \leq C(p) \iint_{\mathbb{R}\times\mathbb{R}} \frac{dxd\xi}{(|\xi|^2 + V(x))^p}.$$

Applying this, we have

$$\sum_{j\geq 1} \lambda_j^{\theta-1} = \mathrm{Tr}[H^{\theta-1}] \leq 2^{1-\theta}\mathrm{Tr}[(H+\lambda_1)^{\theta-1}] \leq C(\theta) \iint_{\mathbb{R}\times\mathbb{R}} \frac{dxd\xi}{(|\xi|^2 + V(x) + \lambda_1)^{1-\theta}}.$$

If $V(x) \geq C|x|^s$, then $\sum_{j\geq 1} \lambda_j^{\theta-1} < \infty$ provided $\theta < \frac{1}{2} - \frac{1}{s}$.

Lemma 1 (Regularity of ρ)

ρ is supported on $\mathcal{H}^\theta(\mathbb{R})$ for any $\theta < \frac{1}{2} - \frac{1}{s}$.

$\to$ When $s \leq 2$, a mass renormalization is needed, i.e.,

$$\int_{\mathbb{R}} : |u|^2 := \int_{\mathbb{R}} (|u|^2 - \mathbb{E}_\rho[|u|^2]) = \int_{\mathbb{R}} |u|^2 - \mathbb{E}_\rho \left[\int_{\mathbb{R}} |u|^2 \right].$$

Integrability of Gaussian measure

- For $\beta > 0$ and $1 < p < \infty$, we can use Khintchine's inequality

$$\mathbb{E}_{\mathbb{P}}\left[\left|\sum_j a_j g_j(\omega)\right|^p\right] \le C(p)\|a_j\|_{\ell^2}^p$$

to estimate $\mathbb{E}_\rho[\|u\|^p_{\mathscr{W}^{\beta,p}}]$. Indeed,

$$\begin{aligned}
\mathbb{E}_\rho[\|u\|^p_{\mathscr{W}^{\beta,p}}] &= \int_{\mathbb{R}} \mathbb{E}_\rho[|H^{\beta/2}u(x)|^p]dx \\
&= \int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}\left[\left|H^{\beta/2}\left(\sum_j \frac{g_j(\omega)}{\sqrt{\lambda_j}}u_j(x)\right)\right|^p\right]dx \\
&= \int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}\left[\left|\sum_j \lambda_j^{\frac{\beta-1}{2}} u_j(x) g_j(\omega)\right|^p\right]dx \\
&\le C(p)\int_{\mathbb{R}} \Big(\underbrace{\sum_j \lambda_j^{\beta-1}|u_j(x)|^2}_{=H^{\beta-1}(x,x)}\Big)^{p/2} dx,
\end{aligned}$$

where $H^{\beta-1}(x,y) = \sum_{j\ge1}\lambda_j^{\beta-1}u_j(x)\overline{u_j}(y)$ is the integral kernel of $H^{\beta-1}$.
$\Rightarrow$ ρ is supported in $\mathscr{W}^{\beta,p}(\mathbb{R})$ as long as

$$H^{\beta-1}(.,.) \in L^{p/2}(\mathbb{R}).$$

Integrability of Gaussian measure

For which values of β and q, we have $H^{\beta-1}(.,.) \in L^q$?

- For 1d torus, $|u_j(x)|^2 = 1$, so for $q \geq 1$,

$$\|H^{\beta-1}(.,.)\|_{L^q(\mathbb{T})} \leq \sum_{j\geq 1} \lambda_j^{\beta-1} < \infty$$

  provided $\beta < \frac{1}{2}$. So $H^{\beta-1}(.,.) \in L^q(\mathbb{T})$ for all $0 \leq \beta < \frac{1}{2}$ and any $1 \leq q \leq \infty$.
- For 1d harmonic potential, we have (**Koch, Tataru '05**)

$$\|u_j\|_{L^p(\mathbb{R})} \lesssim \begin{cases} \lambda_j^{-\frac{1}{6}+\frac{1}{3p}} & \text{if } 2 \leq p < 4, \\ \lambda_j^{-\frac{1}{12}} & \text{if } p \geq 4. \end{cases}$$

  Thus

$$\|H^{\beta-1}(.,.)\|_{L^q(\mathbb{R})} \leq \sum_{j\geq 1} \lambda_j^{\beta-1} \|u_j\|^2_{L^{2q}(\mathbb{R})} \lesssim \begin{cases} \sum_{j\geq 1} \lambda_j^{\beta-1-\frac{1}{3}+\frac{1}{3q}} & \text{if } 1 \leq q < 2, \\ \sum_{j\geq 1} \lambda_j^{\beta-1-\frac{1}{6}} & \text{if } q \geq 2. \end{cases}$$

  Since $\lambda_j \sim j$, we have $H^{\beta-1}(.,.) \in L^q(\mathbb{R})$ for

$$\begin{cases} 0 \leq \beta < \frac{1}{3} - \frac{1}{3q} & \text{if } 1 \leq q < 2, \\ 0 \leq \beta < \frac{1}{6} & \text{if } q \geq 2. \end{cases}$$

Integrability of Gaussian measure

Lemma 2 (Integrability of diagonal integral kernel)

Let $0 \le \beta < \frac{1}{2}$. *Then* $H^{\beta-1}(.,.) \in L^q(\mathbb{R})$ *for any*

$$\max\left\{1, \frac{2}{s(1-2\beta)}\right\} < q \le \infty.$$

Proof:

- $q = \infty$: $H \ge C(1-\partial_x^2) \to H^{\beta-1} \le C(\beta)(1-\partial_x^2)^{\beta-1}$ (operator monotonicity)
  $\to H^{\beta-1}(x,x) \le C(\theta)G(x,x)$, where

  $$G(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{i(x-y)\xi}}{(1+|\xi|^2)^{1-\beta}} d\xi$$

  is the Green function of $(1-\partial_x^2)^{1-\beta}$. Note that $G(x,x) < \infty$ as long as $\beta < \frac{1}{2}$.
- For $1 < q < \infty$, we want to show

  $$\sup_{\substack{\chi \in L^{q'}(\mathbb{R}) \\ \chi \ge 0}} \int_{\mathbb{R}} H^{\beta-1}(x,x)\chi(x)dx \le C\|\chi\|_{L^{q'}(\mathbb{R})}.$$

Integrability of Gaussian measure

We write

$$\int_{\mathbb{R}} H^{\beta-1}(x,x)\chi(x)dx = \mathrm{Tr}[\chi^{1/2}H^{\beta-1}\chi^{1/2}] = \|H^{(\beta-1)/2}\chi^{1/2}\|^2_{\mathfrak{S}^2},$$

where

$$\mathfrak{S}^p = \left\{A : \|A\|_{\mathfrak{S}^p} = \left(\mathrm{Tr}[(A^*A)^{p/2}]\right)^{1/p} < \infty\right\}$$

is the p-th Schatten space. For $\alpha > 0$, we write

$$H^{(\beta-1)/2}\chi^{1/2} = \underbrace{H^{\alpha+(\beta-1)/2}}_{\mathfrak{S}^{2q}}\left(\underbrace{H^{-\alpha}(1-\partial_x^2)^{\alpha}}_{\mathfrak{S}^{\infty}}\right)\underbrace{(1-\partial_x^2)^{-\alpha}\chi^{1/2}}_{\mathfrak{S}^{2q'}}.$$

- $H^{\alpha+(\beta-1)/2} \in \mathfrak{S}^{2q} \to \left(\frac{1-\beta}{2} - \alpha\right)2q > \frac{1}{2} + \frac{1}{s}$.
- $H \geq C(1-\partial_x^2) \to H^{-2\alpha} \leq C^{-2\alpha}(1-\partial_x^2)^{-2\alpha} \to (1-\partial_x^2)^{\alpha}H^{-2\alpha}(1-\partial_x^2)^{\alpha} \leq C^{-2\alpha}$, hence $H^{-\alpha}(1-\partial_x^2)^{\alpha} \in \mathfrak{S}^{\infty}$.
- $(1-\partial_x^2)^{-\alpha}\chi^{1/2} \in \mathfrak{S}^{2q'}$. We use the **Kato–Seiler–Simon** inequality: for $1 \leq p < \infty$,

$$\|f(-i\nabla)g(x)\|_{\mathfrak{S}^p} \leq \|f\|_{L^p(\mathbb{R})}\|g\|_{L^p(\mathbb{R})}.$$

It requires $4\alpha q' > 1$ or $\alpha > \frac{1}{4} - \frac{1}{4q}$.

Normalizability

Lemma 3 (Integrability of Gaussian measure)

ρ is supported on $\mathcal{W}^{\beta,p}(\mathbb{R})$ for any $0 \le \beta < \frac{1}{2}$ and

$$\max\left\{2, \frac{4}{s(1-2\beta)}\right\} < p \le \infty.$$

- **Defocusing** : We want to show

$$Z = \mathbb{E}_\rho\left[\exp\left(-\frac{1}{p}\int_{\mathbb{R}} |u|^p\right)\right] \in (0,\infty).$$

Jensen's inequality gives

$$Z \ge \exp\left(-\frac{1}{p}\mathbb{E}_\rho\left[\int_{\mathbb{R}} |u|^p\right]\right) > 0$$

provided $p > \max\{2, \frac{4}{s}\}$.

Normalizability

▶ **Focusing** : The idea is to use the Boué-Dupuis variational formula as follows. Denote a centered Gaussian process $Y(t)$ by

$$Y(t)=\sum_j \frac{B_j(t)}{\sqrt{\lambda_j}} e_j,$$

where $\{B_j\}_{j\geq 1}$ is a sequence of independent complex-valued Brownian motions, i.e., $B_j(t)\sim \mathcal{N}_{\mathbb{C}}(0,t)$.

Lemma 4 (Boué-Dupuis variational formula)

Fix $\Lambda>0$ and denote P_Λ the projection on E_Λ, i.e., $P_\Lambda u=\sum_{\lambda_j\leq\Lambda}\alpha_j u_j$. Let $F: C^\infty(\mathbb{R})\to\mathbb{R}$ be measurable such that

$$\mathbb{E}_{\mathbb{P}}[|F(P_\Lambda Y(1))|^p]+\mathbb{E}_{\mathbb{P}}[|\exp(-F(P_\Lambda Y(1)))|^q]<\infty$$

for some $1<p,q<\infty$ with $\frac{1}{p}+\frac{1}{q}=1$. Then

$$-\log\mathbb{E}_{\mathbb{P}}[\exp(-F(P_\Lambda Y(1)))]=\inf_{\theta\in\mathbb{H}_a}\mathbb{E}_{\mathbb{P}}\left[F(P_\Lambda Y(1)+P_\Lambda I(\theta)(1))+\frac{1}{2}\int_0^1\|\theta(t)\|^2_{L^2(\mathbb{R})}dt\right],$$

where $\mathbb{H}_a$ is the space of drifts (progressively measurable processes belonging to $L^2([0,1],L^2(\mathbb{R}))$) and

$$I(\theta)(t)=\int_0^t H^{-\frac{1}{2}}\theta(\tau)d\tau.$$

Normalizability

It suffices to prove that

$$\sup_{\Lambda} \mathbb{E}_{\rho}\left[\exp\left(\frac{1}{p}\int_{\mathbb{R}} |P_\Lambda u|^p \cdot \mathbb{1}_{\{|\int_{\mathbb{R}} |P_\Lambda u|^2| \le m\}}\right)\right] < \infty.$$

By monotone convergence theorem, it is enough to prove for $L > 0$,

$$\sup_{\Lambda} \mathbb{E}_{\rho}\left[\exp\left(\min\left\{\frac{1}{p}\int_{\mathbb{R}} |P_\Lambda u|^p, L\right\} \cdot \mathbb{1}_{\{|\int_{\mathbb{R}} |P_\Lambda u|^2| \le m\}}\right)\right] \le C$$

for some constant $C > 0$ independent of L. Since $\mathrm{Law}(Y(1)) = \rho$, it is equivalent to show

$$\sup_{\Lambda} \mathbb{E}_{\mathbb{P}}\left[\exp\left(\min\left\{\frac{1}{p}\int_{\mathbb{R}} |P_\Lambda Y(1)|^p, L\right\} \cdot \mathbb{1}_{\{|\int_{\mathbb{R}} |P_\Lambda Y(1)|^2| \le m\}}\right)\right] \le C$$

for some constant $C > 0$ independent of L. Applying the Boué-Dupuis variational formula to

$$F(P_\Lambda Y(1)) = -\min\left\{\frac{1}{p}\int_{\mathbb{R}} |P_\Lambda Y(1)|^p, L\right\} \cdot \mathbb{1}_{\{|\int_{\mathbb{R}} |P_\Lambda Y(1)|^2| \le m\}},$$

it suffices to **bound from below**

$$\mathbb{E}_{\mathbb{P}}\left[-\min\left\{\frac{1}{p}\int_{\mathbb{R}} |P_\Lambda Y(1) + P_\Lambda I(\theta)(1)|^p, L\right\} \cdot \mathbb{1}_{\{|\int_{\mathbb{R}} |P_\Lambda Y(1)+P_\Lambda I(\theta)(1)|^2| \le m\}} + \frac{1}{2}\int_0^1 \|\theta(t)\|^2_{L^2(\mathbb{R})} dt\right]$$

independent of $\theta \in \mathbb{H}_a$ and independent of Λ, L.

Normalizability

Main ingredients :

- Gagliardo-Nirenberg-Sobolev inequality;
- regularity and integrability of Gaussian measure;
- the pathwise regularity bound

$$\|I(\theta)(1)\|^2_{\mathscr{H}^1(\mathbb{R})} \leq \int_0^1 \|\theta(t)\|^2_{L^2(\mathbb{R})} dt.$$

Invariance of Gibbs measures

The invariance of Gibbs measures follows from **Bourgain's argument '94, '96**:

- **Step 1. Local theory** :
  - When $s > 2$: deterministic LWP in the support of Gibbs measure, i.e., *for* $u_0 \in \mathscr{H}^\theta(\mathbb{R}) \supset supp(\mu)$*, there exists* $\delta \sim \|u_0\|_{\mathscr{H}^\theta}^{-\sigma}$ *with* $\sigma > 0$ *such that solution to (NLS) exists on* $[-\delta, \delta]$.
    Tool: Strichartz estimates with a loss of derivatives due to **Yajima, Zhang '04** .
  - When $s \le 2$: probabilistic LWP in the support of the Gibbs measure, i.e., *there exists* $\Sigma \subset \mathscr{H}^\theta(\mathbb{R}) \supset supp(\mu)$ *with* $\mu(\Sigma) = 1$ *such that for* $u_0 \in \Sigma$*, there exists* $\delta > 0$ *so that solution to (NLS) exists on* $[-\delta, \delta]$*.*
    Idea: to use the integrability of Gaussian measure: For $0 \le \beta < \frac{1}{2}$ and $\max\left\{2, \frac{4}{s(1-2\beta)}\right\} < p \le \infty$, there exist $C, c > 0$ such that
    $$\mathbb{E}_\rho\left[\exp\left(c\|e^{-itH} f\|^2_{\mathscr{W}^{\beta,p}}\right)\right] \le C, \quad \forall t \in \mathbb{R}$$
    to derive **probabilistic Strichartz estimates**: for all $T > 0$, $q \ge 1$ and all $\lambda > 0$,
    $$\rho\left(\|e^{-itH} f\|_{L^q([-T,T],\mathscr{W}^{\beta,p})} > \lambda\right) \le Ce^{-c\frac{\lambda^2}{T^{2/q}}}.$$

Invariance of Gibbs measures

▶ **Step 2. Approximate NLS** : Consider

$$i\partial_t u_\Lambda - Hu_\Lambda = \pm Q_\Lambda(|Q_\Lambda u_\Lambda|^2 Q_\Lambda u_\Lambda),$$

where $Q_\Lambda = \chi(H/\Lambda)$ for a suitable cutoff χ. Decomposition

$$u_\Lambda = u_\Lambda^{\text{low}} + u_\Lambda^{\text{high}}, \quad u_\Lambda^{\text{low}} = P_\Lambda u_\Lambda, \quad u_\Lambda^{\text{high}} = P_\Lambda^\perp u_\Lambda.$$

→ u_Λ exists globally in time.
The approximate measure

$$d\mu_\Lambda(u) = d\mu_\Lambda(u) \otimes d\rho_\Lambda^\perp(u)$$

where

$$d\mu_\Lambda(u) = \frac{1}{Z_\Lambda} \exp\left(\mp\frac{1}{2}\int_{\mathbb{R}} |Q_\Lambda u|^4\right) d\rho_\Lambda(u)$$

is invariant under the flow of the approximate NLS.
→ to use it as a **substitution for the conservation law** to derive a **uniform** (in Λ) estimate for the approximate solutions.

Invariance of Gibbs measures

▶ **Step 3. Estimation of the difference** : to use a PDE approximation argument to estimate the difference between the approximate and the exact solutions.
→ the same uniform estimate holds for the exact solution. Moreover, *for T and $\varepsilon > 0$, there exists $\Sigma_{T,\varepsilon}$ such that*

- $\mu(\Sigma^c_{T,\varepsilon}) < \varepsilon$;
- solution to (NLS) exists on $[-T, T]$ for $u_0 \in \Sigma_{T,\varepsilon}$.

▶ **Step 4. Almost sure GWP and measure invariance** :

- Fix $\varepsilon > 0$ and let $T_n = 2^n$ and $\varepsilon_n = 2^{-n}\varepsilon$. We have the set $\Sigma_n = \Sigma_{T_n,\varepsilon_n}$ as above.
- Let $\Sigma_\varepsilon = \cap_{n=1}^{\infty} \Sigma_n$. Then solution to (NLS) exists globally in time for data in Σ_ε and

$$\mu(\Sigma^c_\varepsilon) = \mu\left(\bigcup_{n=1}^{\infty} \Sigma^c_n\right) \leq \sum_{n=1}^{\infty} \mu(\Sigma^c_n) < \sum_{n=1}^{\infty} 2^{-n}\varepsilon = \varepsilon.$$

- Let $\Sigma = \cup_{\varepsilon>0} \Sigma_\varepsilon$. Then solution to (NLS) exists globally in time for data in Σ and

$$\mu(\Sigma^c) = \mu\left(\bigcap_{\varepsilon>0} \Sigma^c_\varepsilon\right) \leq \inf_{\varepsilon>0} \mu(\Sigma^c_\varepsilon) = 0.$$

From almost sure GWP → measure invariance.

Thank You!