Exploiting rank structures in the cyclic reduction

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Cyclic reduction is a useful tool that can be used for several purposes. It has been originally introduced by G.H. Golub and R.W. Hockney in the mid 1960s with the aim to solve certain linear systems, and has been generalized to solve nonlinear matrix equations defined by matrix power series. A survey of this algorithm and its applications is presented in [1].

An important application concerns the computation of the minimal nonnegative solution of the matrix equation $A + BX + CX^2 = 0$, encountered in Quasi Birth-Death (QBD) Markov chains, where A, B, and C are given $m \times m$ matrices and X is the matrix unknown. The cyclic reduction algorithm relies on the construction of the sequences

$$\begin{aligned} B_{i+1} &= B_i - A_i B_i^{-1} C_i - C_i B_i^{-1} A_i \\ A_{i+1} &= -A_i B_i^{-1} A_i, \qquad\qquad i = 1, 2, \ldots \\ C_{i+1} &= -C_i B_i^{-1} C_i \end{aligned} \tag{1}$$

where $A_1 = A$, $B_1 = B$, $C_1 = C$.

An important instance of this problem, which model double QBDs, is presented in [2] where the matrices A, B and C are tridiagonal and have a very large size. In some cases they are infinite. Under this assumption, the matrices generated by (1) lose their original structure and become dense matrices just after the first step. Apparently, the tridiagonal structure cannot be exploited and the cost of performing (1) becomes very large.

In this work, we analyze the specific case where A, B and C are tridiagonal as in [2] and show that the matrices generated by (1) maintain the quasiseparable structure [3]. Moreover, we provide an analysis of the growth of the quasiseparability rank and show that, numerically, this rank remains bounded independently of the number of iterations.

In order to exploit these properties we implement a reformulation of (1) involving hierarchical matrices.

Finally, we discuss on possible generalizations of this scheme to more general structures.

References

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