Refined BPS number on compact Calabi-Yau 3-folds from Wilson loops

‘Équations différentielles motiviques et au–delà’ in Institute Henri Poincaré

Albrecht Klemm, Bonn University & Sheffield University

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Based on work with

Minxin Huang (ICT Hefei, Anhui), Sheldon Katz (Univ. Urbana, Illinois) and Xin Wang (ICT, Hefei, Anhui): arXiv 2503.16270
Today

Introduction

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of Maldacena, who showed that this expectation value in the $\mathrm{AdS}_4 \times S^5$ conformal gauge string theory correspondence is in leading order of the t'Hooft coupling given as the string world-sheet volume bounding the curve C in the AdS_5 boundary where the conformal $N = 4$ gauge theory lives.

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The geometric engineering program takes local limits X_* of X so that gravity decouples and one ends up with a rigid (non-gravitational) supersymmetric field theory with eight supercharges on X_*.

General strategy:

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The predictions of the latter can be checked mathematically by constructing the the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ Lefshetz decomposition of the cohomology of the moduli space $\hat{M}_\beta$ of stable sheaves F with $\mathrm{ch}_2(F) = \beta$ and $\chi(F) = n$.

Refined BPS invariants in 5d rigid super symmetric theory

M-theory on Calabi-Yau threefold leads to 5d $\mathcal{N} = 1$ supergravity. The BPS states are labelled by their Poaincaré representation defined by their mass and their spin representations w.r.t. the little group

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Using the 4d/5d correspondence by circle compactification we label the latter by the even branes $\mathrm{D}(2K)$-branes $K = 1, \ldots, 3$ of type IIA theory

$$\Gamma = (q_0, q_A, p^A, p^0)$$

The left spin gets identified with the $\mathrm{D0}$ brane charge

$$q_0 = 2\frac{j_L}{(p^0)^2} \ .$$

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$$\sum_{g=0} n_g^\beta I_L^g = \sum_{j_R} (-1)^{2j_R}(2j_R+1) N_{j_L,j_R}^\beta \left[\frac{j_L}{2}\right]_L .$$

with $I_*^n = \left(2[0]_* + \left[\frac{1}{2}\right]_*\right)^{\otimes n} = \sum_{j\geq 0}\left(\binom{2n}{n-j} - \binom{2n}{n-2-j}\right)\left[\frac{j}{2}\right]_*$

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We will see that the $N_{j_L,j_R}^\beta \in \mathbb{N}$ can be seen as dimensions of cohomology groups of $\hat{M}_\beta$ w.r.t. to an $\mathrm{su}(2)_L \times \mathrm{su}(2)_R$ Lefshetz decomposition, while the $n_g^\beta \in \mathbb{Z}$ is an "Euler number" of $\hat{M}_\beta$.

Refined BPS invariants in 5d rigid super symmetric theory

It is well know, see examples below, that in compact Calabi-Yau X the $N^{\beta}_{j_L,j_R} \in \mathbb{N}$ depend on the complex structure of X, while the n^{β}_{g} are truly invariants.

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$$Z_{BPS}(\epsilon_L, \epsilon_R, t) = \mathrm{Tr}_{\mathcal{H}_{BPS}}(-1)^{2(j_L+j_R)} e^{-2\epsilon_L j_L} e^{-2\epsilon_R j_R} e^{-2\epsilon_R j_{\mathcal{R}}} e^{\beta H} \ .$$

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Here the $j_{L/R}$ denote the operators that correspond to the Cartan generators of the $\mathrm{SU}(2)_{L/R}$ above, and $j_{\mathcal{R}}$ denotes the charge operator of an $U(1)_{\mathcal{R}}$ global symmetry, present in the rigid theory.

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Refined BPS invariants in 5d rigid super symmetric theory

Suppose $Z_{BPS}(\epsilon_L, \epsilon_R, H)$ is known to arbitrary orders in $q_{L/R}$ and the Kähler parameter $e^{\beta \cdot t}$, using e.g. the virtual Bialinicki-Birula Birula decomposition of $\mathcal{H}^*(\hat{M}_\beta)$ Choi, Katz, & AK, '12 or the refined topological vertex Iqbal, Kozcaz, Vafa 07.

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Identifying $\mathcal{F}(\epsilon_L, \epsilon_R, t) = \log(Z_{BPS}(\epsilon_L, \epsilon_R, H))$, $H = t$ and reorganizing it in the form suggested by Gopakumar and Vafa

$$\mathcal{F}(\epsilon_L, \epsilon_R, t) = \sum_{\substack{\beta \in H_2(X,\mathbb{Z}) \\ \beta \neq 0}} \sum_{\substack{k=1 \\ j_L, j_R = 0}}^{\infty} (-1)^{2(j_L + j_R)} N^{\beta}_{j_L, j_R} \frac{\chi_{j_L}(q_L^k)\, \chi_{j_R}(q_R^k)}{k\, \mathcal{I}(k\epsilon_1, k\epsilon_2)} e^{k\,\beta\cdot t},$$

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yields the $N^{\beta}_{j_L,j_R}$. Here $\mathcal{I}(\epsilon_1, \epsilon_2) = 2\sinh\left(\frac{\epsilon_1}{2}\right) 2\sinh\left(\frac{\epsilon_2}{2}\right)$ and $\chi_j(x) = \left(\sum_{m=-j}^{j} x^m\right)$.

The refined holomorphic anomaly equations

Let us organize $\mathcal{F}(\epsilon_L, \epsilon_R, t)$ as

$$\mathcal{F}(\epsilon_L, \epsilon_R, t) = \sum_{n,g \in \mathbb{Z}} (\epsilon_1 + \epsilon_2)^{2n} (\epsilon_1 \epsilon_2)^{g-1} F^{(n,g)}(t)$$

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Then the $F^{(n,g)}(t)$ can be calculated using the refined holomorphic anomaly equations

$$\bar{\partial}_{\bar{i}} F^{(n,g)} = \frac{1}{2} \bar{C}_{\bar{i}}^{jk} \left(D_j D_k F^{(n,g-1)} + {\sum_{m,h}}' D_j F^{(m,h)} D_k F^{(n-m,g-h)} \right), n+g > 1\,.$$

in terms of rings of almost holomorphic generators for congruent subgroups. e.g. of $\mathrm{SL}(2,\mathbb{Z})$, for example $\Gamma_1(3)$ for local $\mathbb{P}^2$ $\mathrm{Tot}(\mathcal{O}(-3) \to \mathbb{P}^2)$, Huang, AK '10 and Krefl and Walcher '10 and complete boundary conditions Huang, Kashani-Poor, AK '11.

Representation theoretic aspects

Example 1: Huang, Poretschkin, AK: '13 local del Pezzo Calabi-Yau space $\mathrm{Tot}(\mathcal{O}(-K_S) \to S)$ with $S = d_8\mathbb{P}^2$.

For the BPS states $N^d_{j_L, j_R}$ at $d = 2$ one calculates:

$2j_L \backslash 2j_R$	0	1	2	3
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One sees that the adjoint represention **248** of E_8 appears as the BPS number $N^2_{\frac{1}{2},\frac{3}{2}}$, which decomposes into two Weyl orbits with the weights $w_1 + 8w_0$, further $3876 = \mathbf{1} + \mathbf{3875}$, where the latter decomposes in the Weyl orbits of $w_1 + 7w_8 + 35w_0$.

Representation theoretic aspects

Example 2: Katz, Pandharipande, AK: '14 $S = K3$ For the BPS states $N^d_{j_L,j_R}$ at $d = 3$ one gets:

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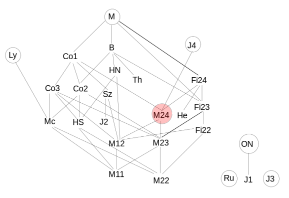

Wilson loops in 5d $\mathcal{N} = 1$ gauge theory

The half-BPS Wilson loop operator located at the origin of $\mathbb{R}^4$ and wound around the S^1 was defined by Young '11, Assel, Estes, Yamazaki '14

$$W_{\mathbf{r}} = \mathrm{Tr}_{\mathbf{r}} \mathcal{T} \left(i \oint_{S^1} dt \, (A_0(t) - \phi(t)) \right),$$

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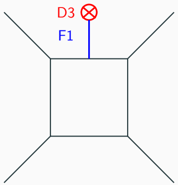
The insertion of the half-BPS Wilson loop operator can be also realized by introducing a half-BPS static, heavy and electrically charged particle at the origin of $\mathbb{R}^4$. We will refer to such a particle as a Wilson loop particle, as explained for $G = \mathrm{SU}(2)$ next.

The Wilson loop particle



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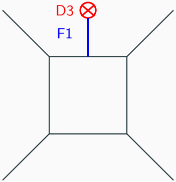

IIB description of the 5d Wilson loops in the $SU(2)$ theory. The (p,q) five-brane web diagram on the left illustrates the (p,q) five-brane configuration in the $x^{5,6}$ directions. It is the dual diagram of the corresponding toric description of the local Calabi-Yau $X = \mathcal{O}(-2,-2) \to \mathbb{P}^1 \times \mathbb{P}^1$.

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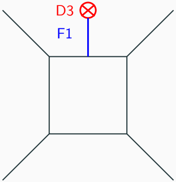

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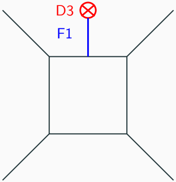

	0	1	2	3	4	5	6	7	8	9
D5	•	•	•	•	•	•				
NS5	•	•	•	•	•		•			
$5_{(p,q)}$	•	•	•	•	•	θ				
F1	•						•			
D3	•							•	•	•

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In the description on the right, the brane configurations are detailed, with $\tan\theta = \frac{p}{q}$ for a five-brane with charge (p, q).

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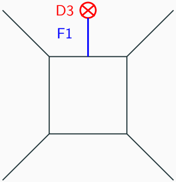

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The half-BPS Wilson loop in the fundamental representation of $SU(2)$ is realized by the fundamental string F1 stretched between a D5 brane and the D3 brane.

Wilson loops in 5d $\mathcal{N} = 1$ gauge theory

The gauge group G is broken to its Abelian subgroup $U(1)^r$ with $r = \mathrm{rank}\, G$ the rank of G. The representation $\mathbf{r}$ becomes the non-negative electric charge $\hat{q}_i$, the gauge charges of the Wilson loop particle under the i-th Abelian gauge subgroup $U(1)$, and is denoted by $\mathbf{r} = [\hat{q}_1, \cdots, \hat{q}_r]$.

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The refined BPS partition functions of 5d theories in the presence of half-BPS Wilson loop operators can be computed as a sum of the k-instanton supersymmetric index of the ADHM quantum mechanics which can be read off from the IIB brane realization Nekrasov '15, Tong and Wong '14, Tong '14

The Wilson loop in 5d $\mathcal{N} = 1$ BPS partition function

Let now X_* a non-compact Calabi Yau space that geometrical engineers a 5d $\mathcal{N} = 1$ theory ($\mathrm{Tot}(\mathcal{O}(-3) \to \mathbb{P}^2)$ works as a further limit) and let $r = b_4^c(X_*)$ be the compact divisors of X_*

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It was proposed in Huang, Lee and Wang '22 that the insertion of a Wilson loop particle with charge $\mathbf{r} = [\hat{q}_1, \cdots, \hat{q}_r]$ can be realized by inserting a wrapped non-dynamic M2-brane over a non-compact curve $\widehat{C}$, ending on boundary $D3$ branes.

The Wilson loop in 5d $\mathcal{N} = 1$ BPS partition function

Let now X_* a non-compact Calabi Yau space that geometrical engineers a 5d $\mathcal{N} = 1$ theory ($\mathrm{Tot}(\mathcal{O}(-3) \to \mathbb{P}^2)$ works as a further limit) and let $r = b_4^c(X_*)$ be the compact divisors of X_*

It was proposed in Huang, Lee and Wang '22 that the insertion of a Wilson loop particle with charge $\mathbf{r} = [\hat{q}_1, \cdots, \hat{q}_r]$ can be realized by inserting a wrapped non-dynamic M2-brane over a non-compact curve $\widehat{C}$, ending on boundary $D3$ branes.

The electric charges of the Wilson loop particles are computed as the intersection numbers of the compact divisors $D_i, i = 1, \cdots, r$, and the non-compact curve $\widehat{C} = \hat{q}_1 \widehat{C}_1 + \cdots + \hat{q}_r \widehat{C}_r$, where

$$D_i \cdot \widehat{C}_j = \delta_{i,j}, \quad i, j = 1, \cdots, r\,,$$

hence $\hat{q}_i = D_i \cdot \widehat{C}, \quad i = 1, \cdots, r$.

The Wilson loop in 5d $\mathcal{N}=1$ BPS partition function

One can arrange the Wilson expectation values in a generating function

$$\begin{aligned} Z_{\text{gen}} &= \exp\left(\sum_{\hat{q}_i \geq 0} \frac{1}{\prod_{i=1}^{r} \hat{q}_i!} \mathcal{F}_{\mathsf{W},\hat{q}}(\epsilon_1, \epsilon_2, t, \hat{t}\,)\right) \\ &= e^{\mathcal{F}(\epsilon_1,\epsilon_2,t)} \left(1 + \sum_{|\hat{q}|>0} \frac{1}{\prod_{i=1}^{r} \hat{q}_i!} \langle W_{\hat{q}} \rangle e^{\hat{q}\cdot\hat{m}}\right), \end{aligned}$$

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with

$$\mathcal{F}_{{\rm W},\hat q}(\epsilon_1,\epsilon_2,t,\hat t)=\mathcal{I}^{|\hat q|-1}\sum_{\beta\in H_2(X;\mathbb{Z})}\sum_{j_L,j_R}(-1)^{2j_L+2j_R}\widetilde{N}^{\beta}_{j_L,j_R}\chi_{j_L}(q_L)\chi_{j_R}(q_R)e^{\beta\cdot t+\hat q\cdot\hat t}$$

where the $\tilde N^{\beta}_{j_R,j_L}$ are the refined BPS numbers in the presence of the Wilson line.

HAE for the 5d $\mathcal{N} = 1$ Wilson line BPS partition function

One can arrange the Wilson partition function as

$$\mathcal{F}_{\mathsf{W},\hat{q}} = \sum_{n,g=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2n} (\epsilon_1 \epsilon_2)^{g+|\hat{q}|-1} \mathcal{F}_{\hat{q}}^{(n,g)}.$$

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The $\mathcal{F}_{\hat{q}}^{(n,g)}$ fullfill a refined holomorphic anomaly equation

$$\bar{\partial}_{\bar{i}} \mathcal{F}_{\hat{q}}^{(n,g)} = \frac{1}{2} \bar{C}_{\bar{i}}^{jk} \left[D_j D_k \mathcal{F}_{\hat{q}}^{(n,g-1)} + \sum_{n',g',\hat{q}'}{}' \prod_{i=1}^{r} \binom{\hat{q}_i}{\hat{q}'_i} D_j \mathcal{F}_{\hat{q}'}^{(n',g')} D_k \mathcal{F}_{\hat{q}-\hat{q}'}^{(n-n',g-g')} \right]$$

Huang, Lee and Wang '22, Wang '23. The latter can be evaluated using a few calculable boundary conditions. This makes the calculation of the $\langle W_{\hat{q}} \rangle$ and the $\tilde{N}^{\beta}_{j_L j_R}$ highly efficient.

Connection to the refined topological partition function on the compact CY X

Consider a 5d $\mathcal{N} = 1$ supergravity theory obtained from M-theory compactified on X. Denote a curve in X as $\widehat{C}$. In the large volume limit of $\widehat{C}$, the volume of X becomes infinite and gravity is decoupled.

Connection to the refined topological partition function on the compact CY X

Consider a 5d $\mathcal{N}=1$ supergravity theory obtained from M-theory compactified on X. Denote a curve in X as $\widehat{C}$. In the large volume limit of $\widehat{C}$, the volume of X becomes infinite and gravity is decoupled.

Under this limit, the half-BPS particles arising from M2 branes wrapping $\widehat{C}$ become heavy and their dynamic degrees of freedom are frozen and we can treat these particles as Wilson loop particles in the local theory.

Connection to the refined topological partition function on the compact CY X

More precisely, let $C, \widehat{C} \in H_2(X; \mathbb{Z})$ denote curves in the compact Calabi-Yau 3-fold X, with Kähler parameters $t, \hat{t}$. In a local limit by taking $\hat{t}$ to $-\infty$, we obtain a non-compact CY3 X_* with Kähler parameters t. Let $D_i \in H^4(X_*; \mathbb{Z})$ represent the compact divisors in X_* and let ϕ_i denote the dual Kähler parameters associated with D_i.

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The gauge charge neutral combination

$$\hat{m} = \hat{t} - \sum_i \hat{q}_i \phi_i, \qquad \hat{q}_i = D_i \cdot \widehat{C},$$

defines the masses for the Wilson loop particles, where the parameters ϕ_i are Coulomb parameters in the effective 5d local theory.

Connection to the refined topological partition function on the compact CY X

In the most general case, we can find a curve class $\widehat{C} \in H_2(X;\mathbb{Z})$ in the compact Calabi-Yau 3-fold X whose large volume limit consists of neighborhoods X_i of mutually disjoint connected compact divisors $D^{(1)}, D^{(2)}, \ldots, D^{(n)}$, each defining a local theory $\mathcal{T}_{X_i}$.

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Decomposing $D^{(i)}$ into its components $D_j^{(i)}$, we have the Wilson loop charges

$$q_j^{(i)} := D_j^{(i)} \cdot \widehat{C} \geq 0, \quad D_j^{(i)} \in H^4(X_i;\mathbb{Z}).$$

The partition function for each local theory $Z_{X_i}(t^{(i)}) = Z_{X_i}(\epsilon_1, \epsilon_2; t^{(i)})$ depends on the Kähler parameters $t_j^{(i)}, j = 1, \cdots, b_2(X_i)$, which can be reformulated in the 5d gauge theory in terms of Coulomb parameters $\phi_j^{(i)}, j = 1, \cdots, r_i$ and mass parameters $m_l^{(i)}, l = 1, \cdots, f_i$ given on the next slide.

Connection to the refined topological partition function on the compact CY X

$$t_j^{(i)} = \sum_{l=1}^{r_i} \phi_l^{(i)} Q_{G,lj}^{(i)} + \sum_{l=1}^{f_i} m_l^{(i)} Q_{F,lj}^{(i)} \,,$$

where $r_i = b_4(X_i)$ is the rank of the gauge group and $f_i = b_2(X_i) - b_4(X_i)$ is the rank of the flavor group.

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$Q_F^{(i)}$ is the intersection matrix of selected non-compact divisors and the compact curves $C_j^{(i)}$. These matrices respectively compute the gauge charges and flavor charges of the BPS particles arising from wrapping an M2-brane on $C_j^{(i)}$.

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We claim that the partition function of the compact CY3 can be written as a linear combination of Wilson loop partition functions in the local theory $\mathcal{T}_{X_i}$ in *all* representations.

Ansatz: Refined partition function on X from Wilson loops on X_*.

The interpretation of the half BPS Wilson line particles as coming from M2-branes on the $\widehat{C}$ direction that become heavy, carry electric charges $\hat{q}^{(i)}$, and become the sources of the half-BPS Wilson loops along the time direction S^1 in each local theory $\mathcal{T}_{X_i}$ suggest the following ansatz for the refined partition function on X

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$$Z_X(t) = \prod_{i=1}^{n} Z_{X_i}(t^{(i)}) \cdot \left[1 + \sum_{k=1}^{\infty} e^{k\hat{m}} Z_k \right],$$

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where $\hat{m} = t_{\widehat{C}} - \sum_{i=1}^{n} \sum_{j=1}^{b_4(X_i)} q_j^{(i)} \phi_j^{(i)}$ is the effective mass for the Wilson loop particle, $\phi_j^{(i)}$ are Kähler parameters with respect to bases in $H_2(X_i; \mathbb{Q})$ that are dual to $D_j^{(i)}$ which are also the Coulomb parameters in each local theory $\mathcal{T}_{X_i}$ and Z_k is a product of linear combinations of Wilson loops.

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The latter satisfy the ansatz

$$Z_k(t) = \prod_{i=1}^{n} \sum_{h_1^{(i)},\cdots,h_{r_i}^{(i)}\geq 0} \sum_{\beta_m^{(i)}\in\mathbb{Z}} P^{(i)}_{[h_1^{(i)},\cdots,h_{r_i}^{(i)}],k;\beta_m^{(i)}} \left\langle W^{(i)}_{[h_1^{(i)},\cdots,h_{r_i}^{(i)}]} \right\rangle e^{\beta_m^{(i)}\cdot m^{(i)}} .$$

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Here we have coefficients which only depend on $e^{\epsilon_1}, e^{\epsilon_2}$ and can be written as

$$\prod_{i=1}^{n} P^{(i)}_{[h_1^{(i)},\cdots,h_{r_i}^{(i)}],k;\beta_m^{(i)}}(\epsilon_1,\epsilon_2) = \frac{\mathcal{P}[e^{\epsilon_1}, e^{\epsilon_2}]}{\prod_{l=1}^{k} 4\sinh(l\epsilon_1/2)\sinh(l\epsilon_2/2)},$$

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The Laurent polynomials $\mathcal{P}[e^{\epsilon_1}, e^{\epsilon_2}]$ are bounded in degree and pol order and the the one-form symmetry. They are in general not completely fixed by this constraints. But they can be determined to certain degree with input of a few for refined BPS numbers, which yields many more concrete predictions.

Constraints from the one form symmetries:

Denote the one-form symmetry of the 5d local theory $\mathcal{T}_{X_i}$ by $\Gamma_i^{(1)} = \prod_j Z_{p_j^{(i)}}$. As pointed out in Morrison et al '20 $\Gamma_i^{(1)}$ can be calculated from the Smith normal form of the charge matrix

$$\mathrm{SNF}(Q_G^{(i)}) = U^{(i)} \cdot Q_G^{(i)} \cdot V^{(i)} = \begin{pmatrix} p_1^{(i)} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & p_2^{(i)} & \cdots & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & & \vdots \\ 0 & \cdots & & p_{r_i}^{(i)} & \cdots & 0 \end{pmatrix},$$

where the main diagonal entries are positive integers satisfying $p_1^{(i)} \leq \cdots \leq p_{r_i}^{(i)}$. $U^{(i)}$ and $V^{(i)}$ are invertible unimodular matrices satisfying $\det U^{(i)} = \det V^{(i)} = 1$, and both their entries and those of their inverse matrices are integers.

Constraints from the one form symmetries:

Let $c_{j,1}^{(i)}, \cdots, c_{j,r_i}^{(i)}$ be the integral one-form symmetry charges for the Wilson loops under $Z_{p_j^{(i)}}$. Then the charge of $\left\langle W^{(i)}_{[h_1^{(i)},\cdots,h_{r_i}^{(i)}]} \right\rangle$ under $\mathbb{Z}_{p_j^{(i)}}$ is $c_{j,1}^{(i)} h_1^{(i)} + \cdots + c_{j,r_i}^{(i)} h_{r_i}^{(i)} \mod p_j^{(i)}$.

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The first constraint on the ansatz is that the Wilson loops must have the same charge under the one-form symmetry $\Gamma^{(i)}$

$$\boxed{c_{j,1}^{(i)} h_1^{(i)} + \cdots + c_{j,r_i}^{(i)} h_{r_i}^{(i)} = k(c_{j,1}^{(i)} q_1^{(i)} + \cdots + c_{j,r_i}^{(i)} q_{r_i}^{(i)}) \mod p_j^{(i)}, \ i = 1, \cdots, n.}$$

Predictions:

Example: From $\mathrm{Tot}(\mathcal{O}(-3) \to \mathbb{P}^2)$ to the elliptic fibration over $\mathbb{P}^2$ called $X_{18}(1,1,1,6,9)$

$2j_L \backslash 2j_R$	0	1	2	3
0		546		
1		1		1

$d_B = 1$

$2j_L \backslash 2j_R$	0	1	2	3	4	5	6
0					546		
1			1		1		1

$d_B = 2$

Table 1: The refined BPS numbers for $d_E = 1$

$2j_L \backslash 2j_R$	0	1	2	3	4	5	6	7	8	9	10
0						546	1	546	1		1
1				1		2		2	546	1	
2							1		1		1

$d_B = 3$

$2j_L \backslash 2j_R$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0					546	1	546	3	1092	5	546	4	546	2	
1			1		2		4	546	5	1092	6	1092	4		2
2						1		3		5	546	5	546	3	
3									1		2		2	546	1
4												1		1	

$d_B = 4$

Table 2: The refined BPS numbers for $d_E = 1$

Dependence on the complex structure of X

An elaborate example for the complex structure dependence of the $N^{\beta}_{j_R,j_L}$ is provided in Section 5.3.2 of the paper. It is based on the simple geometry of smooth ruled surfaces $S \in X$. The projection map $\rho : S \to C$ with $\mathbb{P}^1$ fibres maps to a curve C_g of genus g.

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They occur for example in hypersurfaces $p = 0$ in toric varieties $\mathbb{P}_{\Delta}$ constructed by Batyrev using the 4d reflexive lattice polytope Δ which together with $\hat{\Delta}$ form a dual pair $(\Delta, \hat{\Delta})$.

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They occur for example in hypersurfaces $p = 0$ in toric varieties $\mathbb{P}_\Delta$ constructed by Batyrev using the 4d reflexive lattice polytope Δ which together with $\hat{\Delta}$ form a dual pair $(\Delta, \hat{\Delta})$.

If a codimension two face θ_2 with inner points $l(\theta_2)$ is dual to a codimension three edge $\hat{\theta}_3$ in $\hat{\Delta}$ with inner points $l(\hat{\theta}_3)$ then some monomial deformations of p that in general represent complex structure deformation of X cannot occur, instead there exist $K = l(\theta_2) \cdot l(\hat{\theta}_3)$ additional independent Beltrami differentials μ_k, $k = 1, \ldots K$ in $H^1(\hat{M}, T_{\hat{M}})$ which correspond to *non-polynomial* complex structure deformations.

Dependence on the complex structure of X

The latter are frozen to particular values in the toric embedding of X. If $g = l(\theta_2) \neq 0$ and $l(\hat{\theta}_3) = 1$ then for these frozen values of the moduli a rule surface $S \subset X$ over a genus g curve C_g is realized.

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Let us consider a genus zero curve in the class β represented by a fibre $\mathbb{P}^1$. The moduli space of each $\mathbb{P}^1$ $\hat{M}_\beta$ is identified with C_g and the $\mathrm{SU}(2)_L \times \mathrm{SU}_R(2)$ Lefshetz decompositions yields $R_B^\beta = 2g\,[0,0] + \left[\frac{1}{2}, \frac{1}{2}\right]$.

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If the geometry is deformed w.r.t. to μ_k deformations, the holomorphically embedded curve C_g disappears and the $\mathbb{P}^1$ are fixed to $2g-2$ points, which corresponds to the representation $R_A^\beta = (2g-2)[0,0]$. Clearly the weighted trace over j_R yields the same n_b^β while the $N^\beta_{j_L,j_R}$ change before and after the complex structure deformation.

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More general geometric considerations and checks

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More general mathematical checks are made in the paper and summarised in Sheldon Katz's talk at the Simons Center: https://scgp.stonybrook.edu/video_portal/video.php?id=6922, so I will be brief.

More general geometric considerations and checks

$\hat{M}_\beta$ is the moduli space of stable one dimensional sheaves F on X

$$\mathrm{ch}_2(F) = \beta = (dH \textit{ in local } \mathbb{P}^2 \textit{ example})$$
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$\Phi_\beta = \mathbb{C}[D_\beta]$ is a perverse sheaf of vanishing cycles for $W \equiv 0$.

More general geometric considerations and checks

Then Maulik and Toda, Inv Math '18 define

unrefined BPS invariants by
$\sum_i \chi({}^p R^i \pi_{\beta *} \Phi_\beta) y^i = \sum_g n_g^\beta (y^{\frac{1}{2}} + y^{-\frac{1}{2}})^{2g}$

Note Φ_β can depend on the orientation, which is a choice of the square root of the canonical bundle of $K^{vir}_{\hat{M}_\beta}$ defined line bundle wit fiber

$$\left(K^{\mathrm{vir}}_{\widehat{M}_\beta}\right)_F = \bigotimes_{i=0}^{3} \left(\det \mathrm{Ext}^i(F,F)\right)^{\otimes(-1)^i} \quad .$$

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$$\left(K^{\mathrm{vir}}_{\widehat{M}_\beta}\right)_F = \bigotimes_{i=0}^{3} \left(\det \mathrm{Ext}^i(F,F)\right)^{\otimes(-1)^i} \quad .$$

A canonical orientation always exits Joyce and Upmaier '21

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In this case it agrees with other definitions of $N^\beta_{J_R, j_L}$ mentioned above.

Conclusions

The approach predicts the refined invariants of compact elliptically fibred Calabi-Yau 3 folds X from the local gauge theory limits X_* and passes many mathematical checks.

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If boundary conditions for the ansatz change due to known complex structure dependence of X the additionally predicted BPS numbers pass also consistency checks. This gives better understanding how the refined BPS numbers depend on the complex structure.

Conclusions

The ansatz meets the requirement by the *Completeness Hypothesis* of Polchinski '04, which states that for any gauge theory coupled to gravity, there must exist charged matter in *every* representation of the gauge group, and suggests the breaking of one-form symmetry in the supergravity theory.