Algebraic specialisations of elliptic Gamma functions.

Equations différentielles motiviques, et au-delà, IHP 17/1/2025

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joint work with Nicolas Bergeron and Luis García

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Hilbert's 12th problem (1900)

(Hilbert, transl. Bull. AMS 1902). " It will be seen that in the problem just sketched the three fundamental branches of mathematics, number theory, algebra and function theory, come into closest touch with one another, and I am certain that the theory of analytical functions of several variables in particular would be notably enriched if one should succeed in finding and discussing those functions which play the part for algebraic number field corresponding to that of the exponential function in the field of rational numbers and of the elliptic modular functions in the imaginary quadratic number field."

Very much an open problem. To set the stage for our discussion of the case of complex cubic fields K, we proceed by analogy and first review the positive results that motivated this problem (when $K = \mathbf{Q}$ and K imaginary quadratic).

— I —

Some classical results relevant to Hilbert's 12th problem :

Cyclotomic units, elliptic units

Cyclotomic units : If $(a, f) = 1$ are coprime integers, then

$$u_a(f) = (1 - e^{\frac{2i\pi a}{f}}) = 1 - q_{\frac{a}{f}}$$

is a f-unit in the ring $\mathbf{Z}[e^{\frac{2i\pi}{f}}]$, and a unit as soon as f is divisible by two distinct primes.

It is rather these kind of units that we are after.

Let α be a *modular unit*. It is a nowhere vanishing function on the modular curve $\mathcal{H}/\Gamma_0(N)$, where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \ : \ N \text{ divides } c \right\}.$$

Example. The quotient

$$\Delta(\tau)/\Delta(N\tau) \quad \text{where} \quad \Delta(\tau) = q_\tau \prod_{n=1}^{\infty} (1 - q_\tau^n)^{24},$$

or more generally the expressions

$$\Delta_\delta(\tau) = \prod_{d|N} \Delta(d\tau)^{n_d} \quad \text{with} \quad \sum_{d|N} n_d = 0,$$

are modular units.

Complex Multiplication (CM) theory by Shimura and Taniyama implies that if τ_0 is in $\mathcal{H} \cap K$, with K imaginary quadratic, then $\alpha(\tau_0)$ belongs to an abelian extension of K.

Elliptic units : We consider more generally the theta function on $\mathbf{C} \times \mathcal{H}$

$$\theta_0(z,\tau) = (1 - q_z) \prod_{n=1}^{\infty} (1 - q_\tau^n q_z)(1 - q_\tau^n q_{-z}),$$

Symmetries :

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$$\text{i) periodicities: } \theta_0(z,\tau) = -\theta_0(z+1,\tau) = \underbrace{-e^{2i\pi z}}\theta_0(z+\tau,\tau).$$

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$$\text{ii) distribution : } \quad \theta_0\left(z,\tau\right) = \prod_{k,\ell=1}^{N}\theta_0\left(\frac{z+k+\ell\tau}{N},\tau\right).$$

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iii) modularity under $SL_2(\mathbf{Z})$, a sample :

$$\theta_0(z,\tau+1) = \theta_0(z,\tau),$$

$$\boxed{\theta_0(\frac{z}{\tau}, -\frac{1}{\tau}) = e^{i\pi B_2(z,\tau)}\theta_0(z,\tau)}, \text{ where } B_2 \in \mathbf{Q}(\tau)[z]_{\leq 2}.$$

$$B_2(z,\tau) = \frac{z^2}{\tau} + z(\frac{1}{\tau} - 1) + \frac{\tau}{6} + \frac{1}{6\tau} - \frac{1}{2}.$$

Kronecker's Jugendtraum (1880), CM specialisations

When τ_0 is CM and $\mathfrak{a}$ is an integral ideal of K of norm $N = \mathrm{N}(\mathfrak{a})$ coprime to 6, the ratio

$$\theta_{\mathfrak{a}}(z, \tau_0) = \theta_0(z, \tau_0)^N / \theta_0(Nz, N\tau_0)$$

becomes $\mathbf{Z} + \tau_0 \mathbf{Z}$-periodic in z.
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Let v_0 be a primitive $\mathfrak{f}$-division point in L. If $(\mathfrak{a}, 6\mathfrak{f}) = 1$, then

1. $u_{\mathfrak{a}}(\mathfrak{f}) = \theta_{\mathfrak{a}}(v_0, \tau_0)$ is a $\mathfrak{f}$-**unit** in $K(\mathfrak{f})$, and
2. for every integral ideal $\mathfrak{c}$ coprime to $\mathfrak{f}$,

$$\sigma_{\mathfrak{c}}(u_{\mathfrak{a}}(\mathfrak{f})) = u_{\mathfrak{a}\mathfrak{c}}(\mathfrak{f}) u_{\mathfrak{c}}(\mathfrak{f})^{-\mathrm{N}(\mathfrak{c})}.$$

This essentially solves Hilbert's 12th problem when the base field is imaginary quadratic.

$K_0 = \mathbb{Q}(i), \tau_0 = \frac{2+i}{5}$.

$\mathfrak{a}_0 = 2 + i$ of norm $N_0 = 5$,

with $v_0 = \frac{1}{6}$ a 6-division point ($\mathfrak{f} = 6$).
Then

$$\theta_{\mathfrak{a}_0}(v_0, \tau_0) = \frac{\theta_0(\frac{1}{6}, \frac{2+i}{5})^5}{\theta_0(\frac{5}{6}, 2+i)} \simeq 2.285570413 \cdots - \mathrm{i} \times 1.82956517 \cdots .$$

By CM theory, this complex number is a root of a unitary $P \in \mathbb{Z}[x]_{\leq 8}$ defining a unit in $K_0(6)$.

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$$\texttt{algdep}(\alpha, n, B)$$

(relying on LLL's algorithm) finds a polynomial $P \in \mathbf{Z}_n[X]$ such that $P(\alpha) \approx 0$ up to B digits.

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The **modulus** of these elliptic units encode special values (at $s = 0$) of partial zeta functions of K by means of the Kronecker limit formula.(more on this connection later on).

In this sense they are prototype of Stark units.

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Let $v_0 \in \mathfrak{f}^{-1}L$ be a primitive $\mathfrak{f}$-division point in L. If $(\mathfrak{a}, 6\mathfrak{f}) = 1$, then

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This essentially solves Hilbert's 12th problem when the base field is imaginary quadratic.
Shimura partially generalized this result to CM fields (Drawback : he did not get the construction of units though). Goren-De Shalit 97' paper is a more specific construction of not-quite-units invariants $u(\mathfrak{a}, \mathfrak{b}) \in$ Hilbert class field of a quartic CM field as well.

These were solutions to Hilbert's twelfth problem when the base field is an imaginary quadratic field. Aside from CM fields, Hilbert's 12th problem over <u>other ground fields</u> cannot be approached using the theory of complex multiplication.

Some recent proposals involve replacing complex analytic functions by p-adic analytic ones. For example, Darmon–Dasgupta and Dasgupta give a conjectural p-adic formula for units in abelian extensions of totally real fields.

In their recent breakthrough Dasgupta and Kakde have proved the Brumer–Stark conjecture using p-adic methods. This implies in particular that the conjectural formulas of Darmon-Dasgupta and Dasgupta indeed hold.
A recent (and different) approach by Darmon-Pozzi-Vonk also establishes a related result in the real quadratic case. Both methods involve p-adic deformations of Galois representations.

Beyond SL_2 ? complex cubic field case

From now on, $K \subset \mathbf{C}$ is a complex cubic field,
$\mathcal{O}_K = \mathbf{Z} + \tau\mathbf{Z} + \sigma\mathbf{Z} \simeq \mathbf{Z}^3$ possess a $\mathbf{Z}$-basis $(1, \tau, \sigma)$.

What is the relevant function ? Some kind of infinite product ? ...

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Obstruction : It is classical that there are no meromorphic functions on $\mathbf{C}$ with 3 linearly independent periods.

The beautiful elliptic Gamma function

A major role in this talk will be played by the elliptic Gamma function, a youthful special function introduced in 1997 by Ruijsenaars; it is defined and meromorphic on the domain $\mathbf{C} \times \mathcal{H} \times \mathcal{H}$, where it is given by the c.v. infinite double product

$$\boxed{\Gamma(z,\tau,\sigma) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i((j+1)\tau+(k+1)\sigma-z)}}{1 - e^{2\pi i(j\tau+k\sigma+z)}}}$$

It appears in Baxter's formula for the free energy of the eight–vertex model in statistical physics and hypergeom. solutions to the qKZB equations.

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A recent result of "équations différentielles motiviques" flavor about $\Gamma(z, \tau, \sigma)$ is the following :

Theorem

(M. Kato, 2021). If $(\tau, \sigma) = \omega(q_1, q_2)$,$q_j \in \mathbf{Q}^\times$, *then* $z \mapsto \Gamma(z, \omega(q_1, q_2))$ *satisfies an algebraic differential equation, i.e. given by* $F(z, y, y', \dots y^{(n)}) = 0$ *for some* $F \in \mathbf{C}(z)[Y_0, \dots, Y_n]$.

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(relies on same ppty holding for $\theta(z,\tau)$. Not expected to hold for general τ,σ though).

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It appears in statistical physics, and looks like a higher analog of the Jacobi theta function $\theta_0(z, \tau)$.

Rather than discuss its origin of $\Gamma(z, \tau, \sigma)$ in physics, I'd like to motivate its definition by providing some historical perspective as to why it is indeed a natural candidate to Kronecker/ Hilbert's quest. I'd like to argue that some avatar of this function, and maybe even some of its algebraic properties, might have been anticipated by Eisenstein as early as 1844.

21 years old G. Eisenstein (1844) observes :

$$\sum_{a,b,c \in \mathbf{Z}^3} \frac{1}{(a\omega_1 + b\omega_2 + c\omega_3 + z)^k} \text{ diverges.}$$

To begin with, some familiarity with Eisenstein's toolbox helps us getting some grasp on his oracle to come. As you may know, Eisenstein series were primarily defined by a 2-steps conditionally convergent process
(the "Eisenstein summation procedure") e.g.

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which is, up to an additive constant, nothing but the Weierstrass $\mathfrak{p}(z)$ function attached to the lattice $\mathbf{Z}+\tau\mathbf{Z}$, 30 years before W. introduced it. (see Weil's book).

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Eisenstein's procedure is inductive and rests on a trigonometric identity, a 1-dim. avatar connecting rational fractions to q-series :

$$\lim_{N\to\infty} \sum_{m=-N}^{N} \frac{1}{(\tau+m)} = i\pi + \frac{i\pi}{e^{2i\pi\tau}-1} = i\pi + \frac{i\pi}{q-1}.$$

In his very prospective paper of 1844, young Eisenstein actually emphasizes infinite multiple products rather than series. We follow closely his discussion.

a) Simple infinite product (Euler)

$$\prod_{\lambda\in\mathbb{Z}}^{(\mathrm{eis})}\left(1-\frac{z}{\lambda+B}\right)=\frac{\sin\pi(B-z)}{\sin\pi B}=e^{i\pi z}\left(\frac{1-e^{2i\pi(B-z)}}{1-e^{2i\pi B}}\right).$$

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$\rightsquigarrow$ specializes to elliptic units...(when τ is CM and z, C are torsion points + smoothing)

In the same paper, about the higher degree case he indicates :
"We should consider as an analog of higher degree *quotients of quotients of infinite triple products* of the shape

$$\prod_{\lambda,\lambda',\lambda''} \left(1 - \frac{z}{\lambda + \lambda' A + \lambda'' A'}\right),$$

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"**However no such inconvenience arises if we impose some restrictions on the indices such as inequalities conditions....** ... **There is a large class of such functions that is closely connected to Number Theory....** These functions possess very remarkable properties; they lead to the most beautiful researches, and seem to lie at the crossroads where the most difficult parts of analysis and number theory meet."

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("... number theory, algebra and function theory come into closest touch with one another" would be Hilbert's phrasing 56 years later. Note that Eisenstein seems to have more under his belt, since he is being rather specific).

According to Eisenstein, of particular interest are the inequality conditions arising from "geometric progression". Our thesis is that he had in mind the action of the fundamental unit of a complex cubic field.

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G. Eisenstein (1823-1852), and
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** Jacobi, J. Crelle oct. 1845 : "Mr Eisenstein dit que, par analogie, dans la théorie des intégrales abéliennes, il faudrait considérer *des quotients de quotients*. Mais qu'est-ce que c'est que des *quotients de quotients* ? C'est tout simplement des quotients."

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• This 1844 paper of Eisenstein has been ridiculed by Jacobi, and subsequently has been largely ignored (quoted less than 10 times in the next 150 years). It certainly deserves to be re-evaluated.

• **Our 1st aim :** guided by this predictive (and cryptic) assertion of Eisenstein, we first bridge the gap between quotients of quotients of infinite triple products and the elliptic Gamma function.

Connecting $\Gamma(z, \tau, \sigma)$ with an Eisenstein triple product

Following Eisenstein's method and *two* advices altogether.

Theorem (BCG, 2023) : For $N \geq 2$, consider the smoothed quotient

$$Q_{++} = \prod^{(\mathrm{eis})}_{\lambda \in \mathbf{Z}, \lambda', \lambda'' > 0} \frac{\left(1 - \frac{z}{\lambda + \lambda' A + \lambda'' A' + v}\right)^{N}}{\left(1 - \frac{Nz}{\lambda + N\lambda' A + N\lambda'' A' + Nv}\right)}$$

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Recall from Euler that

$$\prod_{\lambda\in\mathbf{Z}}^{(\mathrm{eis})}\left(1-\frac{z}{\lambda+B}\right) = \underbrace{e^{i\pi z}}\left(\frac{1-e^{2i\pi(B-z)}}{1-e^{2i\pi B}}\right).$$

$\rightsquigarrow$ apply twice Euler's formula, once at (z,v) and once at (Nz,Nv) and then take the quotient.

Connecting $\Gamma(z,\tau,\sigma)$ with an Eisenstein triple product

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$$= \prod_{\lambda',\lambda''>0} \frac{(1-e^{2\pi i(\lambda' A+\lambda'' A'+v-z)})^N(1-e^{2\pi i(N\lambda' A+N\lambda'' A'+Nv)})}{(1-e^{2\pi i(\lambda' A+\lambda'' A'+v)})^N(1-e^{2\pi i(N\lambda' A+N\lambda'' A'+Nv-Nz)})}.$$

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$$Q_{++} = \prod_{\lambda\in\mathbf{Z},\lambda',\lambda''>0}^{(\mathrm{eis})} \frac{\left(1-\frac{z}{\lambda+A\lambda'+A'\lambda''+v}\right)^N}{\left(1-\frac{Nz}{\lambda+NA\lambda'+NA'\lambda''+Nv}\right)}$$

$$= \prod_{\lambda',\lambda''>0} \frac{(1-e^{2\pi i(A\lambda'+A'\lambda''+v-z)})^N(1-e^{2\pi i(NA\lambda'+NA'\lambda''+Nv)})}{(1-e^{2\pi i(A\lambda'+A'\lambda''+v)})^N(1-e^{2\pi i(NA\lambda'+NA'\lambda''+Nv-Nz)})}.$$

Construct Q_{--} similarly, now with $\lambda',\lambda''\in\mathbf{Z}$ satisfying the condition $\lambda',\lambda''\leq 0$. It is then straightforward to derive the following formula for the quotient of quotients

$$\frac{Q_{++}}{Q_{--}} = \frac{\Gamma(z-v,A,A')^N/\Gamma(-v,A,A')^N}{\Gamma(Nz-Nv,NA,NA')/\Gamma(-Nv,NA,NA')}. \qquad (1)$$

The smoothed infinite product

$$\mathbb{C} \times \mathcal{H} \times \mathcal{H} \ni \ (z, \tau, \sigma) \mapsto \frac{\Gamma(z, \tau, \sigma)^N}{\Gamma(Nz, N\tau, N\sigma)}$$

can formally be considered as a higher analog of

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Temporary summary

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We now provide the first numerical evidence supporting our belief that, in the setting of complex cubic fields, it is one of the "transcendental functions" that Hilbert was asking for.

First cubic specialization

Choose

$$K = \mathbf{Q}(\beta) \subset \mathbb{C}, \quad \beta = \sqrt[3]{7} \cdot e^{-\frac{2i\pi}{3}}, \quad \beta^3 = 7.$$

Consider the fractional ideal $\mathbf{Z} + \mathbf{Z}\tau_0 + \mathbf{Z}\sigma_0$, with

$$\tau_0 = \frac{2\beta + \beta^2}{15} \in \overline{\mathcal{H}} \quad \text{and} \quad \sigma_0 = -\frac{2+\beta}{15} \in \mathcal{H}.$$

Together with $v_0 = \frac{1}{3} \in \mathbf{Q}$, we compute the complex number

$$\boxed{\Gamma\left(v_0, \tau_0, \sigma_0\right)^5 \cdot \Gamma\left(5v_0, 5\tau_0, 5\sigma_0\right)^{-1} \approx -4.024029\ldots - i \cdot 41.85595\ldots}$$

...

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coincides (up to 1000 digits at least) with a complex root of the polynomial

$$\begin{aligned} Q = x^6 &+ (6\beta^2 - 14\beta + 2)x^5 + (4\beta^2 - 6\beta + 2)x^4 \\ &+ (106\beta^2 - 152\beta - 103)x^3 + (4\beta^2 - 6\beta + 2)x^2 \\ &+ (6\beta^2 - 14\beta + 2)x + 1. \end{aligned}$$

This polynomial is irreducible over K and its splitting field is a cyclic totally complex abelian extension of degree 6 ramified only at 3.

Example continued

In the example above, with $\beta^3 = 7$, the other roots of Q are:

$$\Gamma\left(\frac{1}{3}, \frac{\beta^2+2\beta+75}{345}, -\frac{\beta+32}{345}\right)^5 \cdot \Gamma\left(\frac{5}{3}, \frac{\beta^2+2\beta+75}{69}, -\frac{\beta+32}{69}\right)^{-1}$$

and

$$\Gamma\left(\frac{1}{3}, \frac{\beta^2+2\beta+15}{150}, -\frac{\beta-43}{150}\right)^5 \cdot \Gamma\left(\frac{5}{3}, \frac{\beta^2+2\beta+15}{30}, -\frac{\beta-43}{30}\right)^{-1}.$$

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The action of the relative Galois group translates into the evaluation at various points in K.

Choosing the specialization $z_0 = 1/3, v_0 = -1/3$, we deduce that the quotient of quotient "à la Eisenstein" Q_{++}/Q_{--} introduced above seems to possess remarkable algebraic properties as well.

More on the elliptic Gamma function

To explain our evaluation procedure and set the stage for our general conjecture, we need to say more about the properties of

$$\Gamma(z,\tau,\sigma) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i((j+1)\tau+(k+1)\sigma-z)}}{1 - e^{2\pi i(j\tau+k\sigma+z)}}$$

A) It has internal symmetry properties (elliptic and modular) that make it a higher dimensional analog of $\theta_0(z,\tau)$.

B) Together with $\theta_0(z,\tau)$, they are the first and second level in a ladder of functions $G_r(z,\tau_1,\ldots,\tau_n)$.
(*more on this in Pierre Morain's talk*)

Automorphic properties of $\Gamma(z, \tau, \sigma)$

Felder–Varchenko (2000) prove that $\Gamma(z, \tau, \sigma)$ extends to $\mathbf{C} \times (\mathbf{C} - \mathbf{R}) \times (\mathbf{C} - \mathbf{R})$ and transforms under $\mathrm{SL}_3(\mathbf{Z})$-action.

- Periodicities: $\Gamma(z + 1, \tau, \sigma) = \Gamma(z, \tau, \sigma)$ and

$$\Gamma(z + \tau, \tau, \sigma) = \theta_0(z, \sigma)\Gamma(z, \tau, \sigma).$$

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- Modularity for $\mathrm{SL}_3(\mathbf{Z}) \curvearrowright \mathbb{P}^2(\mathbf{C})$ (a sample):

$$(*)\ \boxed{\Gamma\left(\frac{z}{x_3}, \frac{x_1}{x_3}, \frac{x_2}{x_3}\right) \Gamma\left(\frac{z}{x_1}, \frac{x_2}{x_1}, \frac{x_3}{x_1}\right) \Gamma\left(\frac{z}{x_2}, \frac{x_3}{x_2}, \frac{x_1}{x_2}\right) = e^{-i\pi B_3(z,x)}}$$

where $B_3 \in \mathbf{Q}(x_1, x_2, x_3)[z]$ is of degree 3 in z.

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Should be thought of as a mult. 1-cocycle identity for $SL_3(\mathbf{Z})$ (splitting a 2-cocycle). To make this more precise, let us recast the classical SL_2 case of θ_0 in cohomological terms.

Recall that

$$\boxed{\theta_0(\frac{z}{\tau}, -\frac{1}{\tau}) = e^{i\pi Q(z,\tau)}\theta_0(z,\tau)}, \text{ where } Q \in \mathbf{Q}(\tau)[z]_{\leq 2},$$

arising from the action of $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbf{Z})$.

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It further holds true that for any $\gamma \in SL_2(\mathbf{Z})$,

$$\boxed{\frac{(\theta_0 \mid \gamma)(z,\tau)}{\theta_0(z,\tau)} = e^{2i\pi Q_\gamma(z,\tau)}}, \text{ where } Q_\gamma(z,\tau) \in \mathbf{Q}(z,\tau).$$

The map $\gamma \mapsto e^{2i\pi Q_\gamma(.)}$ appearing on the RHS is a 1-cocycle for $SL_2(\mathbf{Z})$ that θ_0 splits (or transgresses).

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The RHS is a 1-cocycle for $SL_2(\mathbf{Z})$ that θ_0 splits (or transgresses).

To be able to state what holds true for $SL_3(\mathbf{Z})$ one needs to introduce a family of function $\Gamma_{a,b}(z,x)$ (indexed by pairs of primitive column vectors in $\mathbf{Z}^3$ and defined for $z \in \mathbf{C}$ and x in certain opens $U_a \cap U_b \subset \mathbb{P}^2(\mathbf{C})$.

In the definition for $\Gamma_{a,b}(z,x)$ by Felder-Henriques-Rossi-Zhu (D.M.J. 2008), inequalities are dictated by a, b :
Let $C(a,b) = \{\delta \in (\mathbf{Z}^3)^v, \delta(a) > 0, \delta(b) \leq 0\}$, and $\gamma \in (\mathbf{Z}^3)^v$ a generator of the line $\ker(a) \cap \ker(b)$.
Definition :

$$\Gamma_{a,b}(z,x) = \frac{\prod_{\delta \in C(a,b)/\mathbf{Z}\gamma} \left((1 - \exp(-2i\pi \frac{\delta(x)-z}{\gamma(x)})\right)}{\prod_{\delta \in C(a,b)/\mathbf{Z}\gamma} \left(1 - \exp(2i\pi \frac{\delta(x)-z}{\gamma(x)})\right)}$$

We sum up here the main properties they obtained.

Cohomological $SL_3(\mathbf{Z})$ interpretation

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We sum up here the main properties they obtained.

- For $a = e_1, b = e_2$, $\Gamma_{e_1,e_2}(z,x) = \Gamma\left(\frac{z}{x_3}, \frac{x_1}{x_3}, \frac{x_2}{x_3}\right)$.
- More generally, $\Gamma_{a,b}(z,x)$ is a finite product of various ordinary elliptic Γ's (thus it is meromorphic in z and x).
- Equivariance : $\Gamma_{ga,gb}(z, g^{-t}x) = \Gamma_{a,b}(z,x), \quad \forall g \in SL_3(\mathbf{Z})$.

There is an exact definition for the $\Gamma_{a,b}(z,x)$ that Felder-Henriques-Rossi-Zhu came up with (D.M.J. 2008). We sum up here the main properties they obtained.

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- More generally, $\Gamma_{a,b}(z,x)$ is a finite product of various ordinary elliptic Gamma's (thus it is meromorphic in z and x).
- Equivariance : $\Gamma_{ga,gb}(z, g^{-t}x) = \Gamma_{a,b}(z,x), \quad \forall g \in SL_3(\mathbf{Z})$.
- Cocycle/modular symbol property :

$$\Gamma_{a,b}(z,x)\Gamma_{b,c}(z,x)\Gamma_{c,a}(z,x) = e^{2\pi i P_{abc}(z,x)},$$

where $P_{abc}(z,x) \in \mathbf{Q}(x_1,x_2,x_3)[z]$.

The RHS is a 2-cocycle on $SL_3(\mathbf{Z})$ that the collection of $\Gamma_{a,b}$'s split/transgress.

The statement of our general algebraicity conjecture will involve

- the fundamental unit ϵ of the cubic field K,

- the 1-cocycle/modular symbol $\Gamma_{a,b}$ is then specialised at the pair $(a, \epsilon a)$ for a certain base point "$a \in \mathbf{Z}^3$ " that we now describe.

The statement of our general conjecture will involve

- the fundamental unit ϵ of the cubic field K,
- the 1-cocycle/modular symbol $\Gamma_{a,b}$ is then specialised at the pair $(a, \epsilon a)$ for a certain base point "$a \in \mathbf{Z}^3$ " that we now describe.

Let $\mathfrak{f}$ and $\mathfrak{b}$ be two integral ideals of K with $\mathfrak{f} \neq \mathcal{O}_K$ and $(\mathfrak{f}, \mathfrak{b}) = 1$.

Write $L = \mathfrak{f}\mathfrak{b}^{-1}$ and fix a generator

$$\varepsilon \in \mathcal{O}_K^\times \quad \text{s.t.} \quad \varepsilon - 1_K \in \mathfrak{f} \quad \text{and} \quad \varepsilon_{\mathbf{R}} \in (0,1).$$

Let q be the order of 1_K in K/L.

Let $\mathfrak{a}$ be a degree one prime ideal of O_K such that $(\mathfrak{a}, \mathfrak{b}\mathfrak{f}) = 1$.

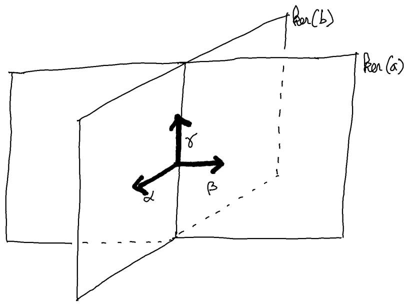
– Lemma 1 – *There exists an oriented* $\mathbf{Z}$*-basis* $(\omega_1, \omega_2, \omega_3)$ *of* $L = \mathfrak{f}\mathfrak{b}^{-1}$ *satisfying*

- $(\omega_1, \omega_2, \omega_3/\mathrm{N}(\mathfrak{a}))$ *is an oriented* $\mathbf{Z}$*-basis of* $\mathfrak{a}^{-1}L$*, and*
- 1_K *is congruent to* ω_3/q *modulo* L*.*

– **Lemma 2** – *there exists a unique primitive element a in $L^v = \mathrm{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$ s.t.*

- $\det(a, \varepsilon a, \cdot) \in L$ *is a positive multiple* $s.\omega_3$ *of* ω_3, *and*
- *the map* $\ker(a) \to \mathbf{C}$ *preserves the orientation induced by the one of* L,
  i.e. $K \hookrightarrow \mathbf{C}$ *defines* $x_0 \in \mathbf{C}^3$ *in the open set* U_a,
  that is $\mathrm{Im}((\lambda \cdot x_0)(\overline{\mu \cdot x_0}) > 0$ *for all oriented bases* (λ, μ) *of* $\ker(a)$.

$\longrightarrow$ *hyperplanes picture (with* $b = \epsilon a$ *and* $\gamma = \omega_3$*)*



If

$$a(\alpha) > 0 \quad \text{and} \quad b(\beta) > 0,$$

then (β, γ) is an oriented basis of ker(a) and (γ, α) is an oriented basis of ker(b).

From now on, we write $b = \varepsilon a$, with $a \in L^v$ primitive chosen so that $\gamma = \omega_3$ from the previous Lemmas.

– **Definition.–** *Let*

$$\boxed{\mathbf{\Gamma}_{\mathfrak{a},\gamma}(L) = \Gamma_{a,b}(\gamma \cdot x_0/q, x_0; \mathfrak{a}^{-1}L) \cdot \Gamma_{a,b}(\gamma \cdot x_0/q, x_0; L)^{-\mathrm{N}(\mathfrak{a})},}$$

with $x_0 \in \mathbf{C}^3$ image of $\omega_1, \omega_2, \omega_3 \in L$ under $K \hookrightarrow \mathbf{C}$.

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– **Conjecture.–** *Suppose that* $(\mathfrak{a}, 6\mathrm{N}(L)) = 1$.

1. *The* $\mathbf{\Gamma}_{\mathfrak{a}}(L)$ *is the image of a unit* $\mathbf{u}_{\mathfrak{a}}(L)$ *in* $K(\mathfrak{f}) \hookrightarrow \mathbf{C}$.
2. *If* w *is an archimedean place above the real place of* K, *then* $|\mathbf{u}_{\mathfrak{a}}(L)|_w = 1$.
3. *For any ideal* $\mathfrak{c}$ *coprime to both* $\mathfrak{f}$ *and* $\mathfrak{a}$, *we have a law of reciprocity:*
$$\sigma_{\mathfrak{c}}(\mathbf{u}_{\mathfrak{a}}(L)) = \mathbf{u}_{\mathfrak{a}}(\mathfrak{c}^{-1}L).$$

— III —

Reasons to believe

Given $\alpha \in \mathbf{C}$, PARI/GP's command

$$\texttt{algdep}(\alpha, n, B)$$

finds a polynomial $P \in \mathbf{Z}_n[X]$ such that $P(\alpha) \approx 0$ up to B digits.

The above 1*st* example was set up for $K = \mathbf{Q}(\beta)$, with $\beta^3 = 7$, together with $\mathfrak{a}$ of norm 5, and $\mathfrak{f}$ of norm 3.

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The above $1st$ example was set up for $K = \mathbf{Q}(\beta)$, with $\beta^3 = 7$, together with $\mathfrak{a}$ of norm 5, and $\mathfrak{f}$ of norm 3. It had the appealing feature that only <u>one</u> ratio of Γ-values was involved. (new kind of "simplest cubic field" problem ?).

We have dozens of other numerical evidences (usually involving many more ratios of Γ-values) for field extensions of cubic fields with absolute degree up to 144. Efficiency/precision of the output of a general algorithm seems like a tricky and important problem.

Recall the modular unit
$\Delta_\delta(\tau) = \prod_{d|N} \Delta(d\tau)^{n_d}$ with $\sum_{d|N} n_d = 0$.

In the imaginary quadratic case, the 1st Kronecker limit formula expresses the real number $|\Delta_\delta(\tau_0)|$ in terms of 1st derivative of partial zeta functions at $s = 0$:

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$$\boxed{\zeta'(\delta, \tau_0, 0) = -\frac{1}{2} \log |\Delta_\delta(\tau_0)|^2}$$

where

$$\zeta(\delta, \tau_0, s) = \sum_{d|N} n_d d^{-s} \zeta_{d\tau_0}(s)$$

and

$$\zeta_{\tau_0}(s) = \sum_{m,n \in \mathbf{Z}}{}' |m\tau_0 + n|^{-2s}.$$

In the complex cubic setting, we consider the partial zeta function

$$\zeta_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},s) = \mathrm{N}(\mathfrak{a}) \sum_{\mathfrak{c}\sim\mathfrak{b}} \mathrm{N}(\mathfrak{c})^{-s} - \sum_{\mathfrak{c}\sim\mathfrak{a}\mathfrak{b}} \mathrm{N}(\mathfrak{c})^{-s}.$$

– Theorem.–(Bergeron-C-Garcia. 2023) *We have unconditionnally*

$$\zeta_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},0)' = \pm \log |\Gamma_{\mathfrak{a}}(\mathfrak{f}\mathfrak{b}^{-1})|^2.$$

The Stark conjecture asserts that $\zeta'_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},0)$ should be equal to $\log|u(\mathfrak{a},\mathfrak{b})|$ for some unit $u(\mathfrak{a},\mathfrak{b}) \in K(f)$. Our conjecture is thus compatible (and it refines) the Stark conjecture in this cubic ATR setting.

II) Connection to other works, ATR number fields

All constructions as above require the ground field K (be it CM, cubic complex) to have exactly one complex embedding. The general setting to formulate these kinds of approaches to Hilbert's 12-th problem that emerges seems to be that of ATR (almost totally real) number fields of arbitrary degree d over $\mathbf{Q}$.

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3. Pierre Morain (Sorbonne Université) on-going PhD work for quartic/quintic... ATR fields K. Uses 2-cocycle for $SL_4(\mathbf{Z})$ and higher dimensional elliptic Gamma functions. Very recent preprint `https://arxiv.org/pdf/2406.06094`

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Thank you for your attention !