Quantum modular forms

Équations différentielles motiviques et au–delà

Campbell Wheeler

18th of October, 2024

Based on joint work with Fantini.

Quantum modular forms

Quantum modular forms f of weight k associated to a multiplicative $\mathrm{SL}_2(\mathbb{Z})$-cocycle Ω satisfy the transformation property

$$f\left(\frac{a\tau+b}{c\tau+d}\right) \;=\; \Omega_\gamma(\tau)\, f(\tau)\, (c\tau+d)^k\,,$$

where $\gamma = [a, b; c, d] \in \mathrm{SL}_2(\mathbb{Z})$. Usual modular forms have $\Omega_\gamma(\tau) = \rho(\gamma)$ for some representation ρ of $\mathrm{SL}_2(\mathbb{Z})$.

Today we will see some natural examples of quantum modular forms f coming from three-manifolds. We will show that their cocycles Ω be equal to the resummation of their asymptotics as τ tends to infinity.

Outline

1. Asymptotics: Quantum invariants of three-manifolds and asymptotics

2. Quantum modular forms: Experimental discoveries to proofs

3. Resurgence: Re-summation and cocycles

Asymptotics: Quantum invariants of three-manifolds

Exponential integrals for 3–manifolds

In the 80s, Jones discovered a remarkable polynomial invariant of links. Witten interpreted the Jones polynomial in terms of quantum field theory. In particular, for $\mathrm{SU}(2)$ connections on a three manifold $\mathcal{A}_M$,

$$Z_M(\hbar) = \int_{\mathcal{A}_M/\mathcal{G}_M} \exp\left(\frac{CS(A)}{2\pi i\hbar}\right) DA$$

where

$$CS(A) = \int_M \mathrm{Tr}(dA \wedge A + \frac{2}{3} A \wedge A \wedge A) \in \mathbb{C} + (2\pi i)^2\mathbb{Z}\,.$$

For $\hbar \in 1/\mathbb{Z}$ Witten related this integral to the invariants of Jones evaluated at certain roots of unity. However if defined, this invariant Z_M is given as an exponential integral over an infinite dimensional space. Therefore, we expect asymptotics as $\hbar \to 0$. The critical points of $CS(A)$ are flat connections,

$$\mathcal{A}^{\mathrm{flat}}_{M,\mathbb{C}}/\mathcal{G}_M \cong \mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})\,.$$

Series at the trivial connection

We can consider the perturbative expansions of these path integrals around the trivial flat connection. These expansions were understood by Feynman diagram techniques by Axelrod-Singer. This leads to invariants call Vassiliev invariants. Kashaev gave a definition using triangulations and Habiro showed that these invariants have a particular form. For a knot K, Habiro showed that Kashaev's invariant is given by

$$J_K(q) \;=\; \sum_{k=0}^{\infty} C_{K,k}(q)(q;q)_k(q^{-1};q^{-1})_k$$

for some $C_{K,k}(q) \in \mathbb{Z}[q^{\pm}]$ and $(x;q)_n = \prod_{j=0}^{n-1}(1-xq^j)$ (where $q = e^{\hbar}$).

Example: 4_1

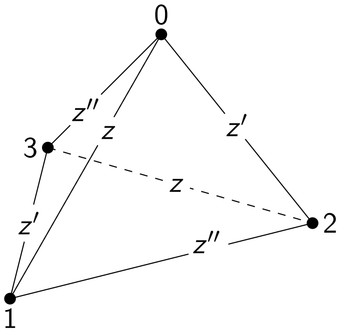
Kashaev showed that $C_{4_1,k}(q) = 1$ so that

$$J_{4_1}(q) \;=\; \sum_{k=0}^{\infty}(q;q)_k(q^{-1};q^{-1})_k\,.$$

$$J_{4_1}(e^{\hbar}) \;=\; 1 - \hbar^2 + \tfrac{47}{12}\hbar^4 - \tfrac{12361}{360}\hbar^6 + \tfrac{10771487}{20160}\hbar^8 + \cdots\,.$$

Ideal tetrahedra and the geometric connection

Thurston used ideal triangulations to construct hyperbolic structures on 3–manifolds. More generally, they can be used to construct flat connections on a three manifold. Hyperbolic ideal tetrahedra are determined by a one dimensional moduli space given by $\mathfrak{h}/(z \mapsto z' \mapsto z'')$ where $z' = 1/(1-z)$ and $z'' = 1 - 1/z$. These symmetries relate to a choice of edge and the angles at the edges of the ideal tetrahedra are determined by the arguments of these numbers.



The volume of such a tetrahedron is given by $D(z) = D(z') = D(z'')$ where $D = \Im(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|$ is the Bloch–Wigner dilogarithm function.

Gluing equations

By lifting a hyperbolic 3–manifold to the universal cover in hyperbolic three space, one can glue together tetrahedra along face using explicit elements of $\mathrm{SL}_2(\mathbb{C})$. These elements are determined by the shapes of the tetrahedra z_i. The vanishing of the monodromy around an edge of the triangulation then leads to algebraic equations of the form

$$\prod_{j=1}^{N} z_j^{A_{ij}} z_j''^{B_{ij}} = \prod_{j=1}^{N} z_j^{A_{ij}} (1 - z_j^{-1})^{B_{ij}} = (-1)^{\nu_i},$$

for some integral A, B, ν. These algebraic equations are called Thurston's gluing equations and the matrices A, B are called Neumann–Zagier data. These matrices satisfy the symplectic properties that

$$AB^t = BA^t, \quad \text{and} \quad (A \,|\, B) \quad \text{is full rank over} \quad \mathbb{Q}.$$

A reduction to finite dimensions

From work of Kashaev in the mid 90s, we expect that Witten's integral should be able to be reduced. In particular, using a triangulation one should be able to express this kind of invariant as a finite dimensional integral where the integrand gets some kind of quantum dilogarithm associated to each tetrahedron. Faddeev's quantum dilogarithm is given for $q = \mathbf{e}(\mathrm{b}^2), \tilde{q} = \mathbf{e}(-\mathrm{b}^{-2})$ with $\Im($

$$\Phi_{\mathrm{b}}(x) \;=\; \frac{(-q^{\frac{1}{2}}\, e^{2\pi \mathrm{b} x}; q)_\infty}{(-\tilde{q}^{\frac{1}{2}}\, e^{2\pi \mathrm{b}^{-1} x}; \tilde{q})_\infty}\,,$$

This has a meromorphic extension to $\tau = \mathrm{b}^2 \in \mathbb{C}\backslash\mathbb{R}_{\leq 0}$. We note that its asymptotics are determined by

$$(ze^{x\hbar^{1/2}}; e^{\hbar})_\infty^{-1} \;\sim\; \widehat{\Psi}_\hbar(x;z) \;=\; \exp\left(-\sum_{k,\ell\in\mathbb{Z}_{\geq 0}} \frac{B_k\, x^\ell\, \hbar^{k+\frac{\ell}{2}-1}}{\ell!\, k!}\mathrm{Li}_{2-k-\ell}(z)\right).$$

These integrals were further explored by Hikami and then formalised by Andersen–Kashaev.

The state integrals

The state integrals are integrals of a Gaussian measure times a product of Faddeev quantum dilogarithms. Explicitly,

$$\int \cdots \int \exp\left(\frac{1}{2}x^t B^{-1} A x/2 + \mu x \mathsf{b} + \nu x \mathsf{b}^{-1}\right) \prod_{j=1}^{N} \Phi_{\mathsf{b}}(x_j) dx$$

where to be an invariant we need to choose an ordered triangulation and use Andersen–Kashaev's choice of contour.

Saddle points

These state integrals are finite dimensional and therefore their asymptotics can be studied via stationary phase. This was studied by Hikami, Dimofte–Gukov–Lenells–Zagier and then formalised by Dimofte–Garoufalidis. This leads to a definition of an asymptotic series for each hyperbolic manifold from the Neumann–Zagier data. In particular, from $M = [A, B, \nu, f, f'', z]$ where $Af + Bf'' = \nu$, $\det(B) \neq 0$ and z satisfies the gluing equations for the geometric connection, Dimofte–Garoufalidis defined

$$\widehat{\Phi}_M(\hbar) \;=\; \left\langle \exp\left(\frac{\hbar^{1/2}}{2} x^t (1 - B^{-1}\nu) + \frac{\hbar}{8} f^t B^{-1} A f\right) \prod_{j=1}^{N} \widehat{\Psi}_\hbar(x_j; z_j) \right\rangle$$

where $\langle\ \rangle$ represents Gaussian integration with respect to the variables x.

Theorem:[Garoufalidis–Storzer–W.,2023]

$\widehat{\Phi}_M(\hbar)$ is a topological invariant of M.

Example: figure eight knot

The NZ datum for 4_1 is given by

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \nu = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad f'' = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and $z_1 = z_2 = \mathbf{e}(1/6)$ and using a Fourier transform identity of the Faddeev quantum dilogarithm we find that

$$\Phi_{4_1}(\hbar) = \left\langle \exp\left(\frac{x}{2}\hbar^{\frac{1}{2}}\right)\Psi_{\hbar}(x, \mathbf{e}(1/6))^2 \right\rangle_{x,\sqrt{-3}}$$

this can then be computed to show

$$\begin{aligned}\Phi_{4_1}(\hbar) = \; & 1 - \frac{11}{216}\sqrt{-3}\,\hbar - \frac{697}{31104}\hbar^2 + \frac{724351}{100776960}\sqrt{-3}\,\hbar^3 + \frac{278392949}{29023764480}\hbar^4 - \frac{244284791741}{43883931893760}\sqrt{-3}\,\hbar^5 \\ & - \frac{1140363907117019}{94789292890521600}\hbar^6 + \frac{212114205337147471}{20474487264352665600}\sqrt{-3}\,\hbar^7 + \frac{367362844229968131557}{117933046642671353856600}\hbar^8 \\ & - \frac{44921192873529779078383921}{126094013470344211542835200 0}\sqrt{-3}\,\hbar^9 - \frac{3174342130562495575602143407}{2310959374147399367912325120 0}\hbar^{10} + O(\hbar^{11})\,.\end{aligned}$$

Neumann–Zagier with no manifold

The asymptotic series just needs $\det(B) \neq 0$ and the critical point around which we expand to be non–degenerate. Therefore, we can define these series more generally for (A, B, ν, f, f'', z) and they will remain invariant under the various moves between these Neumann–Zagier data such as the 2-3 move.

If the quadratic form is degenerate at the critical point, which happens when the critical point is in a component of dimension greater than 0, then more work is needed to define the asymptotic series.

Remark:

Some experiments were done related to examples whose critical points come with positive dimensional components in work with Garoufalidis (Periods, the mero... arXiv:2209.02843). There numerically, asymptotic series were found with coefficients given by periods of the positive dimensional components, in that case the zero locus of the A–polynomial of a knot.

Quantum modular forms: Experimental discoveries to proofs

Original approach to quantum modular forms

The original approach to quantum modular forms is much less precise than the final approach we will take. Here we will describe some of these historical aspects.

A function $f : \mathbb{Q} \to \mathbb{C}$ is a quantum modular form of weight k iff for $\gamma = [a, b; c, d] \in \mathrm{SL}_2(\mathbb{Z})$

$$f(\gamma \cdot \tau) - (c\tau + d)^k f(\tau)$$

is "better" than $f(\tau)$.

Notice that $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{Q} = *$ so straight modularity would be trivial.

Example: Figure 8 knot

The 4_1 or figure 8 knot is one of the simplest hyperbolic knots and has

$$J(q) = J_{4_1}(q) = \sum_{n=0}^{\infty} (-1)^n q^{-n(n+1)/2} (q;q)_n^2.$$

which was our example for something living in the Habiro ring. This was the first really interesting example of quantum modularity (where we take $q = \mathbf{e}(\tau) := \exp(2\pi i \tau)$).

Let $\mathrm{VC} = iV = i\mathrm{Vol}(4_1) = i2.0298\ldots$ (this can be calculated as an exact number using dilogarithms $2D(e(1/6))$ for D the Bloch-Wigner dilogarithm). For $\tau \in \mathbb{Q}$ tending to ∞ (with bounded denominator) Zagier numerically observed

$$\log(J(-1/\tau)) \sim \log(f(\tau)) + V\tau/2\pi + \frac{3}{2}\log(\tau) - \frac{1}{4}\log(3) + \log(\Phi_{4_1}(2\pi i/\tau))$$

for some $f : \mathbb{Q}/\mathbb{Z} \to \mathbb{C}$.

Example: Figure 8 knot

The first few values of the function f were given by

$$f(0) = 1\,, \qquad f(1/2) = 5\,, \qquad f(\pm 1/3) = 13\,, \qquad f(\pm 1/4) = 27\,.$$

These exactly agree with the values $J(1), J(-1), J(\mathbf{e}(\pm 1/3), J(\pm i)$. This is not a coincidence and is true for all τ.

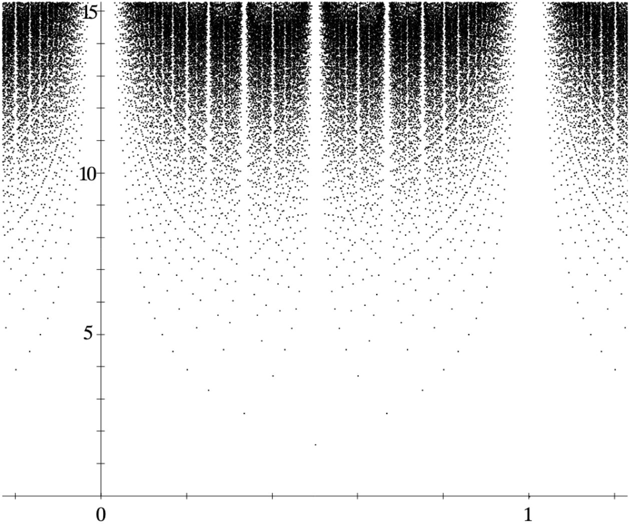
Theorem:[Bettin-Drappeau,Garoufalidis-Zagier]

For $\tau \in \mathbb{Q}$ tending to ∞ (with bounded denominator)

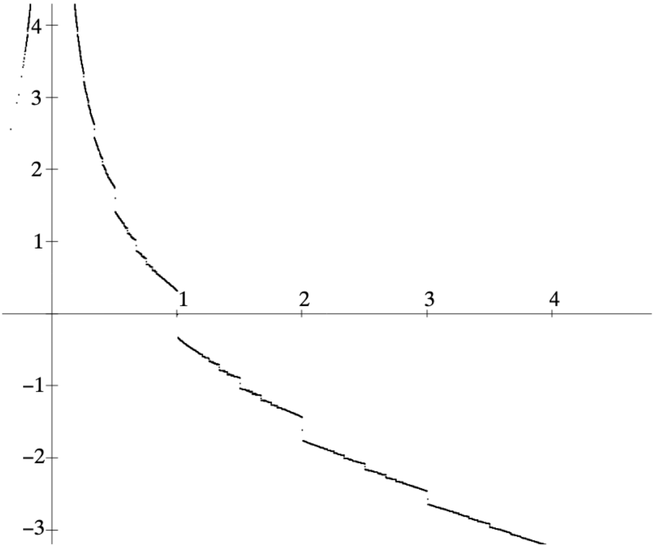
$$\log(J(-1/\tau)) - \log(J(\tau)) \sim V\tau/2\pi + \frac{3}{2}\log(\tau) - \frac{1}{4}\log(3) + \log(\Phi_{4_1}(2\pi i/\tau))$$

The right-hand side is given as an asymptotic series but has some analytic properties as opposed to the original as seen in the following diagram taken from [Quantum modular forms, D. Zagier].

$\log(J(\tau))$



$\log(J(-1/\tau)) - \log(J(\tau))$



Can we do more?

Bettin-Drappeau proved this version of the conjecture for the all hyperbolic knots up to seven crossings (besides one case). Their strategy works for any q-hypergeometric sum however one needs to check that usual asymptotic methods can be applied. The main obstacle is proving that the boundary contributions are negligible and the integral is dominated by the stationary phase approximation around the critical point.

One issue here is that while the failure of modularity is “better” it is far from even being continuous as it is discontinuous at each rational number.

Refining quantum modularity

Since Zagier's original article, Garoufalidis–Zagier have refined the original version of quantum modularity taking into account exponentially small corrections. These can be most easily understood using Borel resummation, which we will discuss later, however they use a smoothed version of optimal truncation related to work of Dingle and Berry.

Let $\hat{\Phi} = 3^{-1/4}\tau^{3/2}\exp(V\tau/2\pi)\Phi_{4_1}(2\pi i/\tau)$. Then the quantum modularity is given by

$$J(-1/x) \sim \hat{\Phi}(2\pi i/x)J(x)\,.$$

Replacing $\hat{\Phi}$ with smooth optimal truncation Garoufalidis-Zagier found that

$$J(-1/x) - \hat{\Phi}^{\mathrm{smop}}(2\pi i/x)J(x) \sim \hat{\Phi}^{\mathrm{new}}(2\pi i/x)J^{\mathrm{new}}(x)\,.$$

This indicated that J should be part of a vector not a single number.

A matrix of invariants

In fact, the function J comes as part of a matrix of similar functions. The rows of the matrix are indexed by a Gröbner basis of an associated q–difference equations while the columns are indexed by objects related to the solutions to the gluing equations (just the points of a 0-dimensional variety).

The matrix $\mathbf{J}$ then satisfies an expression of the form

$$\mathbf{J}(-1/x) \;=\; \Omega(2\pi i/x)\,\mathbf{J}(x)\,\mathbf{j}(x)\,.$$

where Ω is a matrix of extended asymptotic series similar to $\widehat{\Phi}_M$ and $\mathbf{j}(x)$ is an automorphy factor. However, more is expected and we call $\mathbf{J}$ a (matrix valued) quantum modular form when Ω is the restriction of an analytic function on $\mathbb{C} - \mathbb{R}_{\leq 0}$.

One can prove this analytic property using state integrals.

Definition of a quantum modular form

Definition: Analytic cycles

We call $\Omega : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_N\mathcal{O}'$ an analytic cocycle if $\Omega_\gamma(\tau)$ is holomorphic for $\tau \in \mathbb{C}\backslash c^{-1}\mathbb{R}_{\leq -d}$ and satisfies the cocycle condition

$$\Omega_{\gamma_1\gamma_2}(\tau) \;=\; \Omega_{\gamma_1}(\gamma_2\tau)\,\Omega_{\gamma_2}(\tau)\,.$$

Definition: Quantum modular forms

We call $f : \mathbb{Q}(or\ \mathfrak{h}) \to \mathbb{C}^N$ a quantum modular form of weight k with analytic cocycle Ω if it satisfies the transformation property

$$f\Big(\frac{a\tau+b}{c\tau+d}\Big) \;=\; \Omega_\gamma(\tau)\,f(\tau)\,(c\tau+d)^k\,,$$

where $\gamma = [a, b; c, d] \in \mathrm{SL}_2(\mathbb{Z})$.

Usual modular forms have $\Omega_\gamma(\tau) = \rho(\gamma)$ for some representation ρ of $\mathrm{SL}_2(\mathbb{Z})$.

Example: the figure eight knot

Consider the family of functions of $\tau = N/M \in \mathbb{Q}$

$$J_m(q) = \sum_{k=0}^{\infty} (-1)^k q^{-k(k+1)/2+mk} (q;q)_k^2$$

and for $X_j = \frac{1}{2} + (-1)^j \frac{\sqrt{-3}}{2}$

$$J_{i,m}(q) = \sum_{k \in \mathbb{Z}/M\mathbb{Z}} \frac{(-1)^k q^{k(k+1)/2+mk} X_i^{k/M} X_i^{1/2M+m}}{\prod_{j=0}^{N-1} (1 - q^{1+k+j} X_i^{1/M})^{2(1+j+k)/M-1}} .$$

These functions give the matrix

$$\mathbf{J}_m(q) = \begin{pmatrix} 1 & 0 & 0 \\ J_m(q) & J_{m,1}(q) & J_{m,2}(q) \\ J_{m+1}(q) & J_{m+1,1}(q) & J_{m+1,2}(q) \end{pmatrix}$$

We will prove that this is a quantum modular form.

Factorisation of state integrals

The main tool when proving a function is a quantum modular form is factorisation of state integrals. State integrals can often be factorised into products bilinear combinations of functions in $q = \mathbf{e}(\tau)$ and $\tilde{q} = \mathbf{e}(-1/\tau)$ where $\mathbf{e}(x) = \exp(2\pi i x)$ and $\mathrm{b}^2 = \tau$. This was known in the physics literature (for example the work of Beem–Dimofte–Pasquetti) and was explicitly proved for a family of examples by Garoufalidis–Kashaev.

There are two different places one can factorise a state integral. Either when $\tau \in \mathbb{C} - \mathbb{R}$ or when $\tau \in \mathbb{Q}$.

Factorisation at rationals

Factorising at rationals is done using the following lemma of Garoufalidis–Kashaev.

Lemma:[Garoufalidis–Kashaev]

If $U \subseteq \mathbb{C}$ is open, $U + a = U$ and $f : U \to \mathbb{C}$ is an analytic function such that for

$$g(z) = \frac{f(z+a)}{f(z)} \quad \text{we have} \quad g(z+a) = g(z)\,,$$

then if γ is a contour such that $g(z) \neq 1$ on γ then

$$\int_\gamma f(z)\,dz = \Big(\int_\gamma - \int_{\gamma+a} \Big) \frac{f(z)}{1-g(z)}\,dz\,.$$

This has a higher dimensional analogue. We use these formulas and evaluate the state integrals using the residue theorem. A beautiful consequence of this lemma is that for the state integrals the equation $g(z) = 1$ is the the same as the gluing equations (or critical point equations).

Example: figure eight knot

For the state integral of [Garoufalidis–Gu–Mariño–W.],

$$\int_{\mathbb{R}+\epsilon} \Phi_{\mathsf{b}}(x)^2 \frac{\mathbf{e}(-x^2/2)}{1+q^{1/2}\mathbf{e}(-i\mathsf{b}x)}dx$$

we can factorise when $\tau = N/M \in \mathbb{Q}_{>0}$ using the fundamental lemma to find an elementary function times

$$\tau^{3/2}J(q) + \mathbf{e}(V_1/NM(2\pi i)^2)J_1(q)LJ_2(\tilde{q}) + \mathbf{e}(V_2/NM(2\pi i)^2)J_2(q)LJ_1(\tilde{q})$$

where

$$J(q) = \sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2}(q;q)_k^2$$

is the Kashaev invariant of 4_1 and for $X_j = \frac{1}{2} + (-1)^j\frac{\sqrt{-3}}{2}$

$$\color{red}{L}J_i(q) = \sum_{k\in\mathbb{Z}/M\mathbb{Z}} \frac{(-1)^k q^{k(k+1)/2} X_i^{k/M} X_i^{1/2M}}{\color{red}{(1-X_i^{1/M}q^k)}\prod_{j=0}^{N-1}(1-q^{1+k+j}X_i^{1/M})^{2(1+j+k)/M-1}}$$

Proof that $\mathbf{J}(q)$ is a quantum modular form

The proof that $\mathbf{J}(q)$ is a quantum modular form is now essentially done. The state integral

$$\int_{\mathbb{R}+\epsilon} \Phi_{\mathsf{b}}(x)^2 \frac{\mathbf{e}(-x^2/2)}{1+q^{1/2}\mathbf{e}(-i\mathsf{b}x)} dx$$

is analytic for $\tau = \mathsf{b}^2 \in \mathbb{C}\backslash\mathbb{R}_{\leq 0}$ which follows from the same property of Faddeev's quantum dilogarithm. Then the factorisation of the state integral into a sum of bilinear combinations of functions in $q, \tilde{q}$ can be written as an entry of a quotient of matrices

$$\mathbf{J}(-1/x)\,\mathbf{j}(x)^{-1}\mathbf{J}(x)^{-1}\,,$$

which can be proved by q-holonomic methods. This is exactly the statement of being a quantum modular form. The cocycle condition trivially follows as we have written it as the boundary in a larger class of functions.

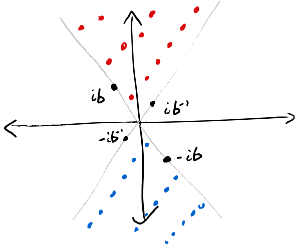
What about q-series

With our success at proving quantum modularity for functions from $\mathbb{Q}$ it is natural to wonder about the case when τ is in the upper half plane $\mathfrak{h}$. Garoufalidis-Zagier by chance discovered q-series that were related to the figure eight knot (with a little help from grep). This remarkable discovery helped lead to many insights into the structure of quantum modular forms. With hindsight, we can now just focus on the state integral and rediscover their q-series by factorisation in the upper half plane.

Factorisation in the upper half plane

One way to factorise a state integral when τ is in the upper half plane is to use the pole structure of the Faddeev quantum dilogarithm. Then one deforms the contour of integration to infinity in a good direction, which reduces the integral to a sum of the residues captured on the way. This sum then decouples i.e. can be made into a bilinear combination of q and $\tilde{q}$ series.

The poles and zeros of the Faddeev quantum dilogarithm are located on cones as can easily be seen from the product formula $\frac{(-q^{\frac{1}{2}}e^{2\pi \mathrm{b}x};q)_\infty}{(-\tilde{q}^{\frac{1}{2}}e^{2\pi \mathrm{b}^{-1}x};\tilde{q})_\infty}$. This is depicted below.



Example: figure eight knot

The state integrals

$$\int_{\mathbb{R}+\epsilon} \Phi_{\mathsf{b}}(x)^2 \mathbf{e}(-x^2/2)dx\,, \quad \text{and} \quad \int_{\mathbb{R}+\epsilon} \Phi_{\mathsf{b}}(x)^2 \frac{\mathbf{e}(-x^2/2)}{1+q^{1/2}\mathbf{e}(-i\mathsf{b}x)}dx$$

factorises as elementary functions times

$$G^{(1)}(q)G^{(0)}(\tilde{q}) - \tau^{-1}G^{(0)}(q)G^{(1)}(\tilde{q})\,, \quad \text{and} \quad G^{(2)}(q) + \tau^{-1}G^{(1)}(q)L^{(0)}(\tilde{q}) - \tau^{-2}G^{(0)}(q)L^{(1)}(\tilde{q})$$

where

$$G^{(0)}(q) = \sum_{n\geq 0}(-1)^n \frac{q^{n(n+1)/2}}{(q;q)_n^2} = 1 - q - 2q^2 - 2q^3 - 2q^4 + \dots$$

$$2G^{(1)}(q) = 2\sum_{n\geq 0}\left(n+1/2-2E_1^{(n)}(q)\right)(-1)^n \frac{q^{n(n+1)/2}}{(q;q)_n^2} = 1 - 7q - 14q^2 - 8q^3 - 2q^4 + \dots$$

$$12G^{(2)}(q) = 12\sum_{n\geq 0}\left(\frac{1}{2}\left(n+1/2-2E_1^{(n)}(q)\right)^2 - E_2^{(n)}(q) - \frac{1}{24}E_2(q)\right)(-1)^n \frac{q^{n(n+1)/2}}{(q;q)_n^2}$$

$$= 1 - 25q - 38q^2 + 58q^3 + 178q^4 + \dots$$

with additional series

$$L^{(0)}(q) = 2E_0^{(1)}(q) + \sum_{n=1}^{\infty}(-1)^n \frac{q^{n(n+1)/2}}{(q;q)_n^2}\frac{q^n}{1-q^n}$$

$$L^{(1)}(q) = \frac{1}{8} - 2E_1^{(0)}(q)^2 - E_2^{(0)}(q) + \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)/2}}{(q)_n^2}\frac{q^n}{1-q^n}\left(n+1/2-2E_1^{(n)}(q)+\frac{1}{1-q^n}\right)$$

Example: the figure eight knot

The series $G^{(0}(q)$ was the one discovered by Garoufalidis–Zagier. We can put these q-series into a matrix as well

$$\mathbf{J}_m(q) = \begin{pmatrix} 1 & 0 & 0 \\ G^{(2)}_m(q) & G^{(0)}_m(q) & G^{(1)}_m(q) \\ G^{(2)}_{m+1}(q) & G^{(0)}_{m+1}(q) & G^{(1)}_{m+1}(q) \end{pmatrix}$$

Combining all of these results we have the following theorem.

Theorem:[Garoufalidis, Gu, Kashaev, Mariño, W., Zagier]

The matrix $\mathbf{J}_m$ of the figure eight knot is a quantum modular form.

Proof: Use the factorisation of the state integrals and a duality of the associated q–difference equations "quadratic relations" to write the entries of Ω as state integrals. Then use the analytic properties of the Faddeev quantum dilogarithm to prove the state integral has similar properties.

Resurgence: Re-summation and cocycles

What happened to the asymptotic series?

Quantum modularity is now easily proved in examples. The question is then how these matrices of analytic functions relate to the asymptotic series. A conjectural answer was given in work of Garoufalidis–Gu–Mariño.

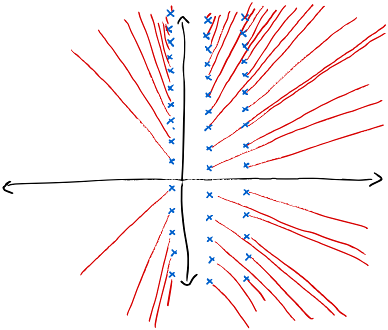
Conjecture:[Garoufalidis–Gu–Mariño]

Combinations of the state integrals associated to a q–hypergeometric sum are equal to the Borel resummation of their asymptotics.

Well firstly one needs that these asymptotic series are Borel resummable which was conjectured by Garoufalidis about ten years prior.

The Borel plane

The structure of the Borel plane for these examples is conjectured to be related to the values of the Chern–Simons invariants V_ρ (of the three manifold or the elements of the Bloch group i.e. solutions to gluing equations). It is conjectured that there are logarithmic branch cuts in Borel plane are at the difference between these values. Given $(V_\rho - V_{\rho'})/(2\pi i)$ is only defined up to $2\pi i$ these branch cuts arrange themselves into peacock patterns.



Stokes phenomenon

Going further, Garoufalidis–Gu–Mariño gave conjectures on the behaviour of the Stokes phenomenon. They conjectured that for each V_ρ there is an collection of asymptotic series with the associated exponential singularity.

[Conjecture: Garoufalidis–Gu–Mariño]

If $\widehat{\Phi}_\rho$ is an asymptotic series with exponential singularity $\mathbf{e}(V_\rho/(2\pi i)^2/\tau)$ then if

$$\arg(\tau) = \arg\left(\frac{V_{\rho'} - V_\rho + (2\pi i)^2 k}{2\pi i}\right)$$

we have

$$s_+(\widehat{\Phi}_\rho)(2\pi i/\tau) - s_-(\widehat{\Phi}_\rho)(2\pi i/\tau) = \sum_{\rho' : \arg(\tau) = \arg(V_{\rho'}/2\pi i)} S_{\rho,\rho',k}\, q^k \widehat{\Phi}_{\rho'}\,,$$

for some $S_{\rho,\rho',k} \in \mathbb{Z}$.

Applications

These conjectures while applied originally to asymptotic series associated to knots in work of Garoufalidis–Gu–Mariño and Garoufalidis–Gu–Mariño–W. can be applied more generally to asymptotic series coming from proper q–hypergeometric functions.

Besides the example of the figure eight knot we can discuss a case where one can carry out computations to get conjectures for generating series of Stokes constants of the q–hypergeometric function

$$\sum_{k=0}^{\infty} \frac{q^{2k^2+mk}}{(q;q)_k}.$$

This will also help illustrate the behaviour of quantum modular forms arising as q-series.

Example: generating series of Stokes constants

The function $\sum_{k=0}^{\infty} \frac{q^{2k^2}}{(q;q)_k}$ has an asymptotic series

$$\widehat{\Phi}(\hbar) \;=\; \exp(V/\hbar)\frac{1}{\sqrt{\delta}}\sum_{k=0}^{\infty} A_k \hbar^k$$

where for $X^4 + X - 1 \;=\; 0$ and

$$V = \mathrm{Li}_2(X) - \frac{\pi^2}{6} + 2(2\pi i)^2 \log(X)^2 - (2\pi i)^2(4k + m(X))\log(X)$$
$$\delta = 4 - 3X\,,$$

with $m(X) \in \mathbb{Z}$ and

$$\begin{aligned}
A_0 &= 1\,,\\
A_1 &= \frac{-64 + 100\,X + 18\,X^2 - 54\,X^3}{24\,\delta^3}\,,\\
A_2 &= \frac{-104876 + 113812\,X + 29836\,X^2 + 17388\,X^3}{1152\,\delta^6}\,,\\
A_3 &= \frac{-79093616 - 1648464240\,X + 2928617760\,X^2 - 694542712\,X^3}{414720\,\delta^9}\,.
\end{aligned}$$

We have four series (one for each embedding of the field into $\mathbb{C}$).

Example: generating series of Stokes constants

We can consider the evaluation of the sum

$$f_m(q) \;=\; \sum_{k=0}^{\infty} \frac{q^{2k^2+km}}{(q;q)_k}\,,$$

at $\tilde{q} = \mathbf{e}(-1/\tau)$ where $\tau = 1000\,\mathbf{e}(0.0001)$ and we find that

$$f_0(\tilde{q}) \;=\; (1.4799\cdots + 1.8058\cdots i) \times 10^{67}\,.$$

we find that the quotient is given by

$$\frac{f_0(\tilde{q})}{s(\widehat{\Phi}^{(3)})(2\pi i/\tau)} \;=\; 1.0000\cdots - 2.7438\cdots \times 10^{-8}\,.$$

Then we see that

$$\Big(\frac{f_0(\tilde{q})}{s(\widehat{\Phi}^{(3)})(2\pi i/\tau)} - 1\Big)q^{-3} \;=\; (1.0197 - 2.4883 \times 10^{-5} \cdot i)\,,$$

and similarly,

$$\Big(\frac{f_0(\tilde{q})}{s(\widehat{\Phi}^{(3)})(2\pi i/\tau)} - 1 - q^3 - q^4 - q^5 - q^6 - q^7 - q^8 - q^9\Big)q^{-10} \;=\; (2.0397\cdots - 5.0718\cdots i \times 10^{-5})\,.$$

Indeed, continuing we can identify this q–series as

$$f_1(q) \;=\; 1 + q^3 + q^4 + q^5 + q^6 + q^7 + q^8 + q^9 + 2q^{10} + 2q^{11} + 3q^{12} + 3q^{13} + \cdots\,,$$

Example: generating series of Stokes constants

Then we find that

$$f_0(\tilde{q}) - s(\widehat{\Phi}^{(3)})(2\pi i/\tau) f_1(q) \;=\; 0.20122\cdots + 0.68776\cdots i\,.$$

Then we can continue this kind of computation to find that numerically

$$\begin{aligned} f_0(\tilde{q}) - s(\widehat{\Phi}^{(1)})(2\pi i/\tau) f_2(q) - s(\widehat{\Phi}^{(2)})(2\pi i/\tau) f_0(q) - s(\widehat{\Phi}^{(3)})(2\pi i/\tau) f_1(q) \\ - s(\widehat{\Phi}^{(3)})(2\pi i/\tau) q f_3(q) \;=\; (5.5399\cdots - 3.7010\cdots i) \times 10^{-138}\,. \end{aligned}$$

This error is exactly the order of numerical error of the Borel resummation. This leads to the conjecture when τ is just above the positive reals

$$\begin{aligned} f_0(\tilde{q}) = {} & s(\widehat{\Phi}^{(1)})(2\pi i/\tau) f_2(q) + s(\widehat{\Phi}^{(2)})(2\pi i/\tau) f_0(q) \\ & + s(\widehat{\Phi}^{(3)})(2\pi i/\tau) f_1(q) + s(\widehat{\Phi}^{(3)})(2\pi i/\tau) q f_3(q)\,. \end{aligned}$$

Performing similar numerical checks for τ just above the negative reals, we find a similar statement

$$\begin{aligned} f_0(\tilde{q}) = {} & s(\widehat{\Phi}^{(1)})(2\pi i/\tau) f_2(q) + s(\widehat{\Phi}^{(2)})(2\pi i/\tau) f_0(q) \\ & + s(\widehat{\Phi}^{(3)})(2\pi i/\tau) q f_3(q) + s(\widehat{\Phi}^{(3)})(2\pi i/\tau) f_1(q)\,. \end{aligned}$$

Example: generating series of Stokes constants

Therefore, we find a q–series when we take the quotient of the two matrices of the Borel resummations that from the conjectures gives generating series for the Stokes constants. In particular, completing f_m to a matrix $F(q)$ (that appears in the factorisation of the state integral)

$$
\begin{aligned}
&s_I(\widehat{\Phi})(\tau)^{-1}s_{II}(\widehat{\Phi})(\tau)\\
&= \begin{pmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&q\end{pmatrix} F(q) \begin{pmatrix}1&0&0&0\\0&\tau&0&0\\0&0&\tau^2&0\\0&0&0&\tau^4\end{pmatrix} F(\tilde{q})^{-1}F(\tilde{q}) \begin{pmatrix}1&0&0&0\\0&-\tau&0&0\\0&0&\tau^2&0\\0&0&0&\tau^4\end{pmatrix}^{-1} F(q)^{-1} \begin{pmatrix}0&0&1&0\\1&0&0&0\\0&0&0&q\\0&1&0&0\end{pmatrix}^{-1}\\
&= \begin{pmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&q\end{pmatrix} F(q) \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} F(q)^{-1} \begin{pmatrix}0&0&1&0\\1&0&0&0\\0&0&0&q\\0&1&0&0\end{pmatrix}^{-1}\\
&= \mathrm{Id} + \begin{pmatrix}-q-2q^2+ & 1+q+q^2+ & 1-q^2+ & -1-q+\\ q^2+ & -q-q^2+ & -q+ & q+q^2+\\ -q-q^2+ & 1-q^2+ & -q-2q^2+ & 2q^2+\\ q+ & -1+q+q^2+ & 2q+q^2+ & -q-2q^2\end{pmatrix} + O(q^3)\,.
\end{aligned}
$$

What is the combination of state integrals?

One of the main issues of Garoufalidis–Gu–Mariño's conjecture is they do not tell us generally what combination of state integrals gives the Borel resummation. In work with Fantini we addressed this issue for the class of formal series coming from the state integrals for $A, B \in \mathbb{Z}_{>0}$ and $A \neq B$

$$\int \exp\left(\frac{1}{2}Ax^2/2 + \mu x\mathsf{b} + \nu x\mathsf{b}^{-1}\right)\Phi_{\mathsf{b}}(x)^B dx\,.$$

To these integrals we can consider the formal Gaussian integration around a critical point of this state integral associated to the each critical point of $(2\pi i)^{-2}B\mathrm{Li}_2(\mathbf{e}(x)) + Ax(x+1)/2$ i.e. the solutions to the equation

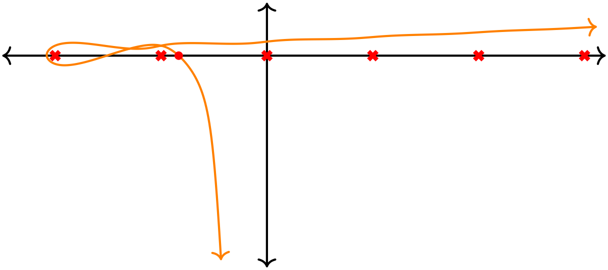
$$(-z)^A \;=\; (1-z)^B\,.$$

Theorem: [Fantini–W.]

There is an explicit algorithm that computes a combination of these state integrals giving the Borel resummation of the associated asymptotic series.

Steepest descent contours

To prove such a statement we attempt to find a deformation of state integral contours where we can apply steepest descent. This would allow us to compute the asymptotics of the state integral along this contour. A steepest descent contour for $(A, B) = (1, 2)$ corresponding to the figure eight knot 4_1 is depicted as follows.

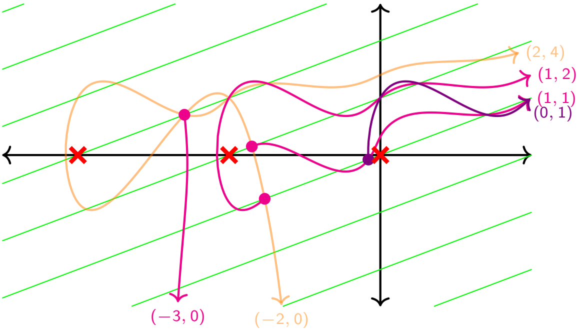


Issue!

The asymptotics of Faddeev's dilogarithm manifests branch cuts in x.

Idea of the algorithm

This means we must break the integrals into further pieces. This requires an inductive procedure introducing more and more paths of steepest descent. One must then prove that this process eventually terminates. A schematic example of this is depicted as follows.



The main theorem

Theorem:[Fantini-W.]

The conjectures of Garoufalidis–Gu–Mariño hold for 4_1 and 5_2 knots. In particular, the asymptotic series associated to these knots by perturbative complex Chern–Simons are Borel re-summable and equal to a combination of state integrals.

Proof: Indeed 4_1 corresponds to $(A, B) = (1, 2)$ and 5_2 corresponds to $(A, B) = (2, 3)$.

Consequences for quantum modularity

Combining this with our previous results on quantum modularity we can prove the following.

Corollary:

The analytic cocycle Ω associated to the matrix $\mathbf{J}_m(q)$ is the Borel re-summation of its asymptotics.

Thanks!

References:

- A TQFT from quantum Teichmüller theory. Andersen, Kashaev.
- Holomorphic Blocks in Three Dimensions. Beem, Dimofte, Pasquetti.
- Modularity and value distribution of quantum invariants of hyperbolic knots. Bettin, Drappeau.
- Quantum content of the gluing equations. Dimofte, Garoufalidis
- Exact Results for Perturbative Chern-Simons Theory with Complex Gauge Group. Dimofte, Gukov, Lenells, Zagier.
- Summability for state integrals of hyperbolic knots. Fantini, W..
- Quantum dilogarithm. Faddeev, Kashaev.
- The resurgent structure of quantum knot invariants. Garoufalidis, Gu, Mariño.
- From state integrals to q–series. Garoufalidis, Kashaev.
- Evaluation of state integrals at rational points. Garoufalidis, Kashaev.
- Resurgence of Chern-Simons theory at the trivial flat connection. Garoufalidis, Gu, Mariño, W..
- Topological invariance of complex Chern-Simons and Teichmüller TQFT perturbation theory. Garoufalidis, Storzer, W..
- Periods, the meromorphic 3D-index and the Turaev–Viro invariant. Garoufalidis, W..
- Knots, perturbative series and quantum modularity Garoufalidis, Zagier.
- Knots and their related q–series. Garoufalidis, Zagier.
- Hyperbolic Structure Arising from a Knot Invariant. Hikami.
- Modular q–difference equations and quantum invariants of hyperbolic three–manifolds. W..
- Quantum exponential function. Woronowicz.
- Quantum field theory and the Jones polynomial. Witten
- Quantum modular forms. Zagier.