Algebraic specialisations of multiple elliptic Gamma functions

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January 2025

Équations différentielles motiviques et au-delà, IHP, Paris

Preprint on arXiv:2406.06094v2

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Hilbert's 12th problem

Theorem (Kronecker-Weber)

$$\overline{\mathbb{Q}}^{ab} = \bigcup_{m \geq 3} \mathbb{Q}(e^{2i\pi/m})$$

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Hilbert's 12th problem

Give an explicit description of the abelian extensions of a given field using analytic functions.

Hilbert's 12th problem: imaginary quadratic case

Imaginary quadratic case: Complex multiplication

If $\mathcal{O}_{\mathbb{K}} = [1, \tau]$ then

$$\overline{\mathbb{K}}^{ab} = \bigcup_{m \geq 2} \mathbb{K}(j(\tau), w(1/m, \tau))$$

where j is the j-invariant and w is Weber's function.

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Elliptic units (Robert)

Abelian extensions of an imaginary quadratic field $\mathbb{K}$ are given by elliptic units built from Jacobi's θ function:

$$\theta(1/q, \tau)^{12N} / \theta(N/q, N\tau)^{12}$$

for explicit $q, N \in \mathbb{Z}$, $\tau \in \mathbb{K}$.

Rank one abelian Stark conjectures

Consider an abelian extension $\mathbb{L}/\mathbb{K}$ in the following cases:

- $\mathbb{K}$ is totally real; only one infinite place v splits completely in $\mathbb{L}$.
- $\mathbb{K}$ has exactly one complex place v and $\mathbb{L}$ is totally complex.

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Rank one abelian Stark conjectures

Fix a place w of $\mathbb{L}$ above v. There is a unit $u \in \mathbb{L}$ such that for all $\sigma \in Gal(\mathbb{L}/\mathbb{K})$,

$$\zeta'(\sigma, 0) = -\frac{1}{e} \log |u^\sigma|_w$$

and such that $\mathbb{L}(u^{1/e})/\mathbb{K}$ is abelian.

Totally Real Fields (case TR_p)

- $\mathbb{K}$ is totally real with a finite place v which splits completely in the totally complex field $\mathbb{L}$

Theorem (Dasgupta, Kakde, 2023)

Abelian extensions of a totally real field $\mathbb{F}$ are given by the Brumer-Stark p-units for which there is an explicit analytic p-adic formula, together with the square root function. In other words,

$$\overline{\mathbb{F}}^{ab} = \bigcup_{\substack{\text{Brumer-Stark units} \\ \text{and conjectures}}} \mathbb{F}(u, \sqrt{\alpha_1}, \ldots, \sqrt{\alpha_{n-1}})$$

where $\alpha_1, \ldots, \alpha_r$ are elements of $\mathbb{F}$ representing all possible signs in $\{-1, 1\}^n/(-1, \ldots, -1)$.

Second Kronecker limit formula

If $\tau = x + iy$ is a CM point and $Q(u,v) = y^{-1}(u+\tau)(v+\overline{\tau})$ is the associated quadratic form, define the associated twisted ζ function for $\Re(s) > 1$ and $(u,v) \notin \mathbb{Z}^2$:

$$\zeta_Q(s,u,v) = \sum_{m,n \in \mathbb{Z}^2 - \{(0,0)\}} e^{2i\pi(mu+nv)} Q(m,n)^{-s}$$

$$\theta_1(z,\tau) = \sum_{n \in \mathbb{Z}} \exp(i\pi((n+1/2)^2\tau + (2n+1)(z-1/2)))$$

Second Kronecker limit formula

$$\zeta_Q(1,u,v) = -\pi \log \left| e^{i\pi\tau u^2} \theta_1(v - u\tau, \tau)/\eta(\tau) \right|^2$$

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The θ and Γ functions

Jacobi's θ function

$$\theta(z,\tau) = \prod_{m\geq 0}(1 - e^{2i\pi(m+1)\tau}e^{-2i\pi z})(1 - e^{2i\pi m\tau}e^{2i\pi z})$$

It is well defined for $\Im(\tau) > 0$ and $z \in \mathbb{C}$. Elliptic units are built from this function and give the Stark units above imaginary quadratic fields.

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Ruijsenaars' elliptic Γ function (1997)

$$\Gamma(z,\tau,\sigma) = \prod_{m,n\geq 0} \frac{1 - e^{2i\pi(m+1)\tau} e^{2i\pi(n+1)\sigma} e^{-2i\pi z}}{1 - e^{2i\pi m\tau} e^{2i\pi n\sigma} e^{2i\pi z}}$$

It is well defined for $\Im(\tau) > 0, \Im(\sigma) > 0$ and $z \notin \mathbb{Z} + \mathbb{Z}_{\leq 0}\tau + \mathbb{Z}_{\leq 0}\sigma$.

Multiple elliptic Gamma functions

Nishizawa's G_r functions (2001)

$$G_r(z,\tau_0,\ldots,\tau_r) = \prod_{m_0,\ldots,m_r\geq 0}\left(1-\prod_{j=0}^{r} e^{2i\pi(m_j+1)\tau_j}e^{-2i\pi z}\right) \times \left(1-\prod_{j=0}^{r} e^{2i\pi m_j\tau_j}e^{2i\pi z}\right)^{(-1)^r}$$

These functions are well-defined if $\Im(\tau_j) > 0$ for all $0 \leq j \leq r$ and for $z \notin \mathbb{Z} + \sum_{j=0}^{r}\mathbb{Z}\tau_j$ if r is odd.

We identify $\theta = G_0$ and $\Gamma = G_1$.

Simple properties

Periodicity:

$$G_r(z, \tau_0, \ldots, \tau_r) = G_r(z+1, \tau_0, \ldots, \tau_r) = G_r(z, \tau_0, \ldots, \tau_j + 1, \ldots, \tau_r)$$

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Pseudo-periodicity:

$$G_r(z+\tau_j, \tau_0, \ldots, \tau_r) = G_{r-1}(z, \tau_0, \ldots, \tau_{j-1}, \tau_{j+1}, \ldots, \tau_r) G_r(z, \tau_0, \ldots, \tau_r)$$

Simple properties

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Example

$$\Gamma(z+\tau,\tau,\sigma) = \theta(z,\sigma)\Gamma(z,\tau,\sigma)$$

$$G_2(z+\tau_0,\tau_0,\tau_1,\tau_2) = G_1(z,\tau_1,\tau_2)G_2(z,\tau_0,\tau_1,\tau_2)$$

Modular property ?

We want an analogue of the modular property:

$$\theta\left(\frac{z}{\tau}, \frac{-1}{\tau}\right) = \exp(i\pi P_0(z,\tau))\theta(z,\tau)$$

where

$$P_0(z,\tau) = \frac{z^2+z}{\tau} - z + \frac{\tau}{6} + \frac{1}{6\tau} - \frac{1}{2}$$

Bernoulli polynomials

Let $\omega_1, \ldots, \omega_d \in \mathbb{C} - \{0\}$. Then put

$$\sum_{n\geq 0} B^*_{d,n}(z, \omega_1, \ldots, \omega_d)\frac{t^n}{n!} = e^{zt}\prod_{j=1}^{d}\frac{\omega_j t}{e^{\omega_j t} - 1}$$

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$$\sum_{n \geq 0} B^*_{d,n}(z, \omega_1, \ldots, \omega_d) \frac{t^n}{n!} = \left(\sum_{m \geq 0} \frac{z^m t^m}{m!} \right) \prod_{1 \leq j \leq d} \left(\sum_{k_j \geq 0} B_{k_j} \omega_j^{k_j} \frac{t^{k_j}}{k_j!} \right)$$

where $B^*_{d,n}(z, \omega_1, \ldots, \omega_d)$ is a homogeneous polynomial of degree n in $d+1$ variables with rational coefficients.

Modular property for G_r

Theorem (Narukawa, 2003)

Consider a family $\omega_1, \ldots, \omega_{r+2} \in \mathbb{C} - \{0\}$ such that for all $j \neq n$, $\omega_j / \omega_n \notin \mathbb{R}$. Then the following equality

$$\prod_{j=1}^{r+2} G_r\left(\frac{z}{\omega_j}, \left(\frac{\omega_n}{\omega_j}\right)_{n\neq j}\right) = \exp\left(\frac{-2i\pi}{(r+2)!}\frac{B^*_{r+2,r+2}(z,\omega_1,\ldots,\omega_{r+2})}{\omega_1\omega_2\ldots\omega_{r+2}}\right)$$

holds whenever the left-hand side makes sense as a function of z.

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Geometric construction

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$$\det_B(a_0, \ldots, a_r, \cdot) \neq 0$$

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- Primitive linear forms $a_0, \ldots, a_r \in \Lambda$ such that

$$\det_B(a_0, \ldots, a_r, \cdot) \neq 0$$

- Define $\gamma \in L$ the unique primitive vector such that:

$$\exists\, s \in \mathbb{Z}_{>0},\ \det_B(a_0, \ldots, a_r, \cdot) = s\gamma$$

- Choose any linear form $c \in \Lambda$ such that $\det_B(a_0, \ldots, a_r, c) = s$.

Geometric construction

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- Define $d_j = sign(\Im(\beta_j))$ and $D = \sum_{j=0}^{r}(d_j - 1)/2$.
- Define $C^+ = \{\delta \in L, a_j(\delta) > 0 \text{ if } d_j = 1, a_j(\delta) \leq 0 \text{ if } d_j = -1\}$.
- Define $C^- = \{\delta \in L, a_j(\delta) \leq 0 \text{ if } d_j = 1, a_j(\delta) > 0 \text{ if } d_j = -1\}$.

Geometric construction

Definition

$$G_{r,a_0,\ldots,a_r}(w,\sigma,L)^{(-1)^D} = \prod_{\delta \in C^{\pm}} \left(1 - e^{\pm 2i\pi\left(\frac{\sigma(\delta)-w}{\sigma(\gamma)}\right)}\right)^{(\pm 1)^r}$$

There are explicitly computable parameters $\tau_0, \ldots, \tau_r \in \sigma(L \otimes \mathbb{Q})$ and an explicit finite set $F_{a_0,\ldots,a_r} = F \subset \mathbb{C}$ such that

$$G_{r,a_0,\ldots,a_r}(w,\sigma,L) = \prod_{z \in F} G_r(z, \tau_0, \ldots, \tau_r)$$

Modular property and equivariance

Theorem (M. 2024)

Let $a_0, \ldots, a_{r+1}$ be linear forms in Λ which are linearly independent. There is an explicit Bernoulli polynomial $B_{r+2,a_0,\ldots,a_{r+1}}(w, \sigma, L)$ such that

1. Modular property:

$$\prod_{j=0}^{r+1} G_{r,(a_k)_{k\neq j}}(w, \sigma, L)^{(-1)^j} = \exp(i\pi B_{r+2,a_0,\ldots,a_{r+1}}(w, \sigma, L))$$

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2. Equivariance property: $\forall\, g \in \mathrm{SL}_{r+2}(\mathbb{Z})$,

$$G_{r,g\cdot a_0,\ldots,g\cdot a_r}(w, g\cdot\sigma, L) = G_{r,a_0,\ldots,a_r}(w,\sigma,L)$$

$$B_{r+2,g\cdot a_0,\ldots,g\cdot a_{r+1}}(w, g\cdot\sigma, L) = B_{r+2,a_0,\ldots,a_{r+1}}(w,\sigma,L)$$

Cocycle relation for $\mathrm{SL}_{r+2}(\mathbb{Z})$:

Fix a primitive vector a in Λ. Consider the following function:

$$\psi_a(g_1, \ldots, g_r) = G_{r,(a, g_1 a, \ldots, (\prod_{j=1}^{r} g_j) a)} = G_{r,[1|g_1|\ldots|g_r]a}$$

where

$$[g_1|\ldots|g_l] = \left(g_1, \ldots, \prod_{j=1}^{k} g_j, \ldots, \prod_{j=1}^{l} g_j \right)$$

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Proposition

The coboundary of ψ_a is:

$$d\psi_a(g_1, \ldots, g_{r+1}) = \exp(i\pi B_{r+2,[1|g_1|\ldots|g_{r+1}]a})$$

Arithmetic construction

- $\mathbb{K}$ an ATR field of degree $d = r + 2 \geq 3$.
- $\mathfrak{f} \neq (1)$ an integral ideal. Set $q\mathbb{Z} = \mathbb{Z} \cap \mathfrak{f}$.
- $\varepsilon_1, \ldots, \varepsilon_r$ fundamental units for $\mathcal{O}_{\mathfrak{f}}^{+,\times}$.

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- $\varepsilon_1, \ldots, \varepsilon_r$ fundamental units for $\mathcal{O}_{\mathfrak{f}}^{+,\times}$.
- $\mathfrak{a}$ a smoothing ideal of prime norm N.
- $\mathfrak{b}$ integral ideals representing classes in $Cl^+(\mathfrak{f})$.
- Fix $L = \mathfrak{f}\mathfrak{b}^{-1}$.
- Fix σ a complex embedding of $\mathbb{K}$.

Arithmetic construction

(Weak) admissibility conditions :

- $h \in L$
- $h/q \equiv 1 \mod L$
- h/N is a generator of the cyclic group $L/(\mathfrak{a}^{-1}L)$

For $\rho \in \mathfrak{S}_r$, for (weakly) admissible vectors $h_\rho \in L$ and orientations $\mu_\rho = \pm 1$, define:

$$u_{\rho,j} = \prod_{i=1}^{j} \varepsilon_{\rho(i)}$$

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A primitive linear form $a_\rho \in \Lambda = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ such that

$$\lambda_\rho a_\rho = \mu_\rho \det_B(h_\rho, u_{\rho,1} h_\rho, \ldots, u_{\rho,r} h_\rho, \cdot), \qquad \lambda_\rho \in \mathbb{Z}_{>0}$$

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$$I_{r,\mathfrak{f},\mathfrak{a},\mathfrak{b}}(\varepsilon_1, \dots, \varepsilon_r, (h_\rho), (\mu_\rho)) = \prod_{\rho \in \mathfrak{S}_r} \frac{G_{r,a_\rho,u_{\rho,1}a_\rho,\dots,u_{\rho,r}a_\rho}(\sigma(h_\rho)/q, \sigma, L)^N}{G_{r,a_\rho,u_{\rho,1}a_\rho,\dots,u_{\rho,r}a_\rho}(\sigma(h_\rho)/q, \sigma, \mathfrak{a}^{-1}L)}$$

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A conjectural Kronecker limit formula for higher degree

Conjecture

Under some conditions on $\mathfrak{f}$ and $\varepsilon_1, \dots, \varepsilon_r$, for any integral ideal $\mathfrak{b}$ representing a class in $Cl^+(\mathfrak{f})$, for any permutation $\rho \in \mathfrak{S}_r$, there is an explicitly computable set of (strongly) admissible vectors $h_{\mathfrak{b},\rho}$ and signs $\mu_{\mathfrak{b},\rho}$ and an explicit finite set $\mathcal{Z}_{\mathfrak{f}}^1$ depending only on $\mathfrak{f}$ such that:

$$u_{\mathfrak{b}} = \prod_{z \in \mathcal{Z}_{\mathfrak{f}}^1} I_{r,\mathfrak{f},\mathfrak{a},\mathfrak{b}}(\varepsilon_1, \dots, \varepsilon_r, (zh_{\mathfrak{b},\rho}), (\mu_{\mathfrak{b},\rho}))$$

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$$u_{\mathfrak{b}} = \prod_{z \in \mathcal{Z}^1_{\mathfrak{f}}} I_{r,\mathfrak{f},\mathfrak{a},\mathfrak{b}}(\varepsilon_1, \ldots, \varepsilon_r, (zh_{\mathfrak{b},\rho}), (\mu_{\mathfrak{b},\rho}))$$

is the image in $\mathbb{C}$ of a unit inside $\mathbb{K}^+(\mathfrak{f})$ related to the Stark unit and

$$N\zeta'_{\mathfrak{f}}([\mathfrak{b}], 0) - \zeta'_{\mathfrak{f}}([\mathfrak{a}\mathfrak{b}], 0) = \frac{1}{\#\mathcal{Z}^1_{\mathfrak{f}}} \log |u_{\mathfrak{b}}|^2$$

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Computations of the G_r functions

$$G_r(z, \tau_0, \ldots, \tau_r) = \begin{cases} \exp\left(\sum_{j\geq 1} \frac{1}{(2i)^r j} \frac{\sin(\pi j(2z-(\tau_0+\cdots+\tau_r)))}{\prod_{k=0}^r \sin(\pi j \tau_k)}\right) & \text{if } r \text{ is odd} \\ \exp\left(\sum_{j\geq 1} \frac{2}{(2i)^{r+1} j} \frac{\cos(\pi j(2z-(\tau_0+\cdots+\tau_r)))}{\prod_{k=0}^r \sin(\pi j \tau_k)}\right) & \text{if } r \text{ is even} \end{cases}$$

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This is convergent for $\tau_j \notin \mathbb{R}$ and $|\Im(2z - \sum \tau_j)| < \sum |\Im(\tau_j)|$.

```
\p 300
x = exp(log(2)/3)*exp(2*I*Pi/3) \\root of x^3 - 2
t = (x*(x+1) + 3)/15
s = (x - 8)/15
f(z, n, j) = sin(Pi*j*n*(2*z-t-s))/(j*sin(Pi*j*t*n)*sin(Pi*j*s*n))
g(z, n) = suminf(j = 1, n*f(z, 1, j) - f(z, n, j))
exp(-g(-1/3, 5)/(2*I))
```

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algdep(%, 6)
```

$x^6 + 3x^5 + 6x^4 + 5x^3 + 6x^2 + 3x + 1$

A quartic example

- Fix $z \in \mathbb{C}$ such that $z^4 - 6z^3 - z^2 - 3z + 1 = 0$ and $\Im(z) > 0$.
- Fix $\mathbb{K} = \mathbb{Q}(z)$, $\mathfrak{f}$ the degree one prime above 2.
- Fix $\mathfrak{a}$ the degree one prime above 13.
- Fix

$$\varepsilon_1 = (-2z^3 + 13z^2 - z + 3)/7, \quad \varepsilon_2 = (-5z^3 + 29z^2 + 15z + 18)/7$$

A quartic example

$\tau = \varepsilon_2 - 15$	$\tau' = -6 + 1/\varepsilon_1 + 1/(\varepsilon_1\varepsilon_2)$
$\sigma = -7 + 1/\varepsilon_2$	$\sigma = -\varepsilon_2 + 15$
$\rho = -\varepsilon_1 - 3$	$\rho' = 4\varepsilon_1 + 19 - 1/\varepsilon_2$

$$\frac{G_2\left(\frac{-1}{2}, \frac{\tau}{26}, \frac{\sigma}{26}, \frac{\rho}{26}\right)^{-13}}{G_2\left(\frac{-13}{2}, \frac{\tau}{2}, \frac{\sigma}{2}, \frac{\rho}{2}\right)^{-1}} \times \frac{G_2\left(\frac{1}{2}, \frac{\tau'}{26}, \frac{\sigma'}{26}, \frac{\rho'}{26}\right)^{13}}{G_2\left(\frac{13}{2}, \frac{\tau'}{2}, \frac{\sigma'}{2}, \frac{\rho'}{2}\right)}$$

A quartic example

$$\frac{G_2\left(\frac{-1}{2},\frac{\tau}{26},\frac{\sigma}{26},\frac{\rho}{26}\right)^{-13}}{G_2\left(\frac{-13}{2},\frac{\tau}{2},\frac{\sigma}{2},\frac{\rho}{2}\right)^{-1}}\times\frac{G_2\left(\frac{1}{2},\frac{\tau'}{26},\frac{\sigma'}{26},\frac{\rho'}{26}\right)^{13}}{G_2\left(\frac{13}{2},\frac{\tau'}{2},\frac{\sigma'}{2},\frac{\rho'}{2}\right)}$$

$$\approx 4.1210208\ldots - i\cdot 5.0617720\ldots$$

We compute 1000 digitis and obtain a root of the polynomial $y^8-7y^7+33y^6+49y^5+17y^4+49y^3+33y^2-7y+1$.

Quintic example

- Starting from the base field given by $x^5 - x^4 - x^3 - 2x^2 + x + 1$

Quintic example

- Starting from the base field given by $x^5 - x^4 - x^3 - 2x^2 + x + 1$
- We build using similar ideas the classfield given by

$$(y^{10}+1)+24(y^9+y)+164(y^8+y^2)+99(y^7+y^3)-62(y^6+y^4)-89y^5$$

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  $x^6 - x^5 - 6x^4 - 13x^3 - 8x^2 + 6x + 3$

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- We build using similar ideas the classfield given by

$$
\begin{aligned}
& y^{12} + 1 - 16081532175162(y^{11} + y) \\
& + 174467692474122866754346581(y^{10} + y^2) \\
& - 491736942447747944014748686(y^9 + y^3) \\
& + 1044987444198027077770817370(y^8 + y^4) \\
& - 1535869228950388545317759682(y^7 + y^5) \\
& + 1737975534353196401914779597y^6
\end{aligned}
$$

Thank you!