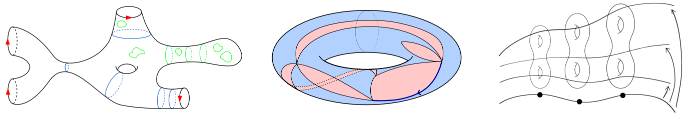
# Topological recursion and its ramifications

**Elba Garcia-Failde**

Sorbonne Université (Institut de Mathématiques de Jussieu - Paris Rive Gauche)



Équations différentielles motiviques et au-delà

Institut Henri Poincaré, May 3rd, 2024

# Outline

1. Topological recursion and its ramifications
   - Introduction, connections, examples
   - One of our playgrounds: Witten–Kontsevich and Airy
2. The quantisation problem
   - Quantum curve problem
   - Origins, context and examples
   - (Classical) spectral curves
3. From loop equations to KZ equations
   - Perturbative wave functions
   - KZ-like equations
   - Bad monodromies
4. Non-perturbative wave functions and Lax system
   - Good monodromies
   - Lax systems
5. Present and future
6. 3 applications: critical exponents, non-perturbative JT, free probability

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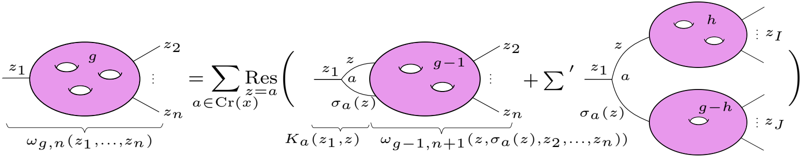
# Topological recursion (TR, Chekhov–Eynard–Orantin '04-'07)

**Goal:** "Count surfaces $S_{g,n}$ of genus $g$ with $n$ boundaries (topology $(g,n)$)."

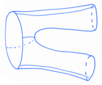
**Spectral curve**

**TR :** $\begin{cases} \Sigma \text{ Riemann surface} \\ x\colon \Sigma \to \mathbb{CP}^1 \\ \omega_{0,1} = y\,dx \text{ 1-form (discs)} \\ \omega_{0,2} \text{ bidifferential (cylinders)} \end{cases}$ $\rightsquigarrow$ **recursion on** $|\chi(S_{g,n})| = 2g-2+n$

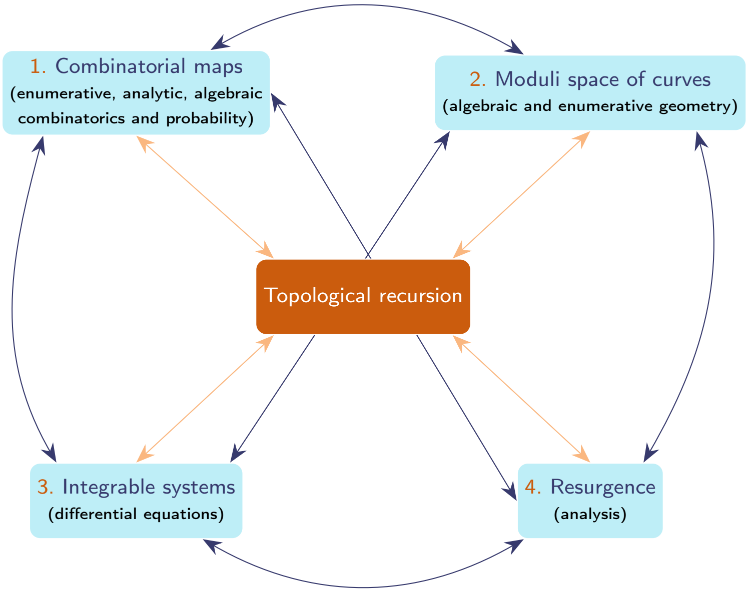
Multi-differentials
$\omega_{g,n}(z_1,\dots,z_n)$, $z_i \in \Sigma$,
$\in H^0(\Sigma^n, K_\Sigma^{\boxtimes n}(\mathcal{P}))^{\mathfrak{S}_n}$




- Terms in correspondence with the ways of cutting a **pair of pants** $(0,3)$ from $S_{g,n}$.



# Connections

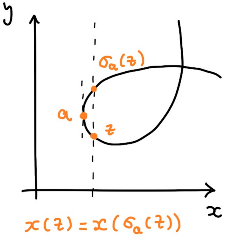


# Spectral curve (input)

**Input:** $\Sigma$ Riemann surface ((global) *spectral curve*),

- $x : \Sigma \to \mathbb{C}$ meromorphic function with finitely many and simple critical/ramification points:
$$\mathrm{Cr}(x) = \mathrm{Ram}(x) := \{p \in \Sigma \mid \mathrm{d}x(p) = 0\},$$
- $\omega_{0,1}$ is a meromorphic 1-form on $\Sigma$, often $\omega_{0,1} = y\mathrm{d}x$, with $y : \Sigma \to \mathbb{C}$ holomorphic on a neighborhood of every $a \in \mathrm{Cr}(x)$ and $\mathrm{d}y(a) \neq 0$,
- $\omega_{0,2}$ is a symmetric bifferential on $\Sigma \times \Sigma$ with only double poles along the diagonal and vanishing residues, that is locally
$$\omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2} + \overbrace{h(z_1, z_2)}^{\text{holomorphic}}.$$

- Around every $a \in \mathrm{Ram}(x)$, $\exists$ local coordinate $\zeta_a(z)$ such that
$$x(z) - x(a) = \zeta_a(z)^2.$$
- $\exists$ neighbourhoods $U_a$ of every $a \in \mathrm{Ram}(x)$ and $\sigma_a : U_a \to U_a$, with
$$x(z) = x(\sigma_a(z)), \forall z \in U_a,$$
$$\sigma_a \neq \mathrm{id}, \sigma_a(a) = a,\ \zeta_a(\sigma_a(z)) = -\zeta_a(z).$$



# Fundamental bidifferentials/Bergman kernel $\omega_{0,2}$

**Fundamental bidifferentials:**

$$\mathcal{B}(\Sigma) := \{H^0(\Sigma^2, K_\Sigma^{\boxtimes 2}(2\Delta))^{\mathfrak{S}_2} \mid \mathrm{res}(B) = 0, \mathrm{bires}(B) = 1\},$$

with $\Delta := \{(z,z) \in \Sigma^2\}$. It is an affine space over

$$H^0(\Sigma^2, K_\Sigma^{\boxtimes 2})^{\mathfrak{S}_2} = \mathcal{S}^2 H^0(\Sigma, K_\Sigma),$$

whose $\dim = \frac{\hat{g}(\hat{g}+1)}{2}$, with $\hat{g} = \mathrm{genus}(\Sigma)$.

**Normalisation:** Symplectic basis $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^{\hat{g}}$ of $H_1(\Sigma, \mathbb{Z})$, $\mathcal{L} := \mathrm{Vect}(\mathcal{A}_1, \dots, \mathcal{A}_{\hat{g}})$ Lagrangian. $\omega_{0,2} = B = B^{\mathcal{L}} \in \mathcal{B}(\Sigma)$ is **normalised** wrt $\mathcal{L}$ if

$$\forall i \in [\![1, \hat{g}]\!] \,, \oint_{z_1 \in \mathcal{A}_i} B^{\mathcal{L}}(z_1, z_2) = 0, \ \forall z_2 \in \Sigma.$$

Then it is **unique**!

**History:** Bergman–Schiffer 50's, Korotkin–Kokotov 00's

Classical Bergman kernel $\rightsquigarrow$ $\frac{\partial^2}{\partial z_1 \partial \overline{z_2}} G(z_1, z_2) \mathrm{d}z_1 \mathrm{d}z_2 = \tilde{B}(z_1, z_2)$

TR Bergman kernel $\rightsquigarrow$ $\frac{\partial^2}{\partial z_1 \partial z_2} G(z_1, z_2) \mathrm{d}z_1 \mathrm{d}z_2 = B(z_1, z_2)$

$G$ Green function

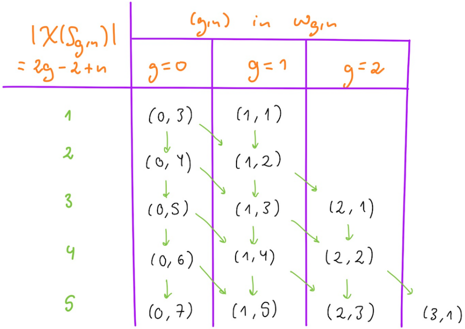
# Multi-differentials (output)

**TR output:** For $n \geq 1$,

$$\omega_{g,n}(z_1,\ldots,z_n) = \sum_{a\in \mathrm{Ram}(x)} \operatorname*{Res}_{z=a} K_a(z_1,z)\Big(\omega_{g-1,n+1}(z,\sigma_a(z),z_{[\![2,n]\!]}) +$$

$$+ \sum_{\substack{0\leq h\leq g \\ I\sqcup J=[\![2,n]\!]}}^{\text{no disk}} \omega_{g-h,1+|I|}(z,z_I)\,\omega_{h,1+|J|}(\sigma_a(z),z_J)\Big),$$

where $K_a(z_1,z) := \frac{\int_{\sigma_a(z)}^{z}\omega_{0,2}(z_1,\cdot)}{2(\omega_{0,1}(z)-\omega_{0,1}(\sigma_a(z)))}$.




$\omega_{g,n}$ multi-differentials, i.e. meromorphic sections of the bundle

$$\bigotimes_{i=1}^{n}\pi_i^*(T^*\Sigma) = \boxtimes_{i=1}^{n} T^*\Sigma$$
$$\downarrow$$
$$\Sigma^n$$

with $\pi_i : \Sigma^n \to \Sigma$ the $i$th projection

# Properties

- $\omega_{g,n} \in H^0(\Sigma^n, K_\Sigma^{\boxtimes n}(\mathcal{P}))^{\mathfrak{S}_n}$, i.e. symmetric with poles at

$$\mathcal{P} = \begin{cases} \{\text{poles of } \omega_{0,1}\}, & \text{if } (g,n) = (0,1), \\ \Delta, & \text{if } (g,n) = (0,2), \\ \mathrm{Ram}(x), & \text{if } 2g-2+n > 0. \end{cases}$$

- **Deformations:** Let $\mathcal{S}_t$ be a family of spectral curves depending on a parameter $t$.

$$\partial_t \omega_{0,1}(z_1) = \int_{\gamma_t} \omega_{0,2}(z, z_1) \text{ (defines a suitable } \gamma_t\text{)},$$

$$\partial_t \omega_{0,2}(z_1, z_2) = \int_{\gamma_t} \omega_{0,3}(z, z_1, z_2),$$

$$\Rightarrow \partial_t \omega_{g,n}(z_1, \dots, z_n) = \int_{\gamma_t} \omega_{g,n+1}(z, z_1, \dots, z_n).$$

- **Dilaton** equation:

$$\sum_{a \in \mathrm{Ram}(x)} \operatorname*{Res}_{z=a} \Phi(z) \omega_{g,n+1}(z_1, \dots, z_n, z) = (2g-2+n)\omega_{g,n}(z_1, \dots, z_m),$$

  where $\mathrm{d}\Phi = \omega_{0,1}$.

- $\omega_{g,n}$ satisfy certain symplectic invariance, loop equations, have modularity properties, are connected to integrable systems...

# Motivations to study TR

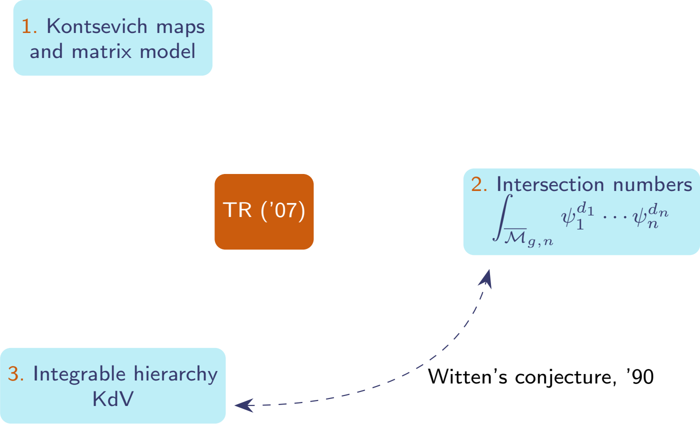
- Allows to calculate
- Gives structure
- Provides universality
- Gaining context
- Nowadays new instances of TR also enrich TR

# Examples

- For the Lambert curve $x = ye^{-y}$, TR provides simple Hurwitz numbers (Eynard–Mulase–Safnuk, '09, arXiv:0907.5224). Hurwitz theory (Bonzom–Chapuy–Charbonnier–G-F, '22, arXiv:2206.14768)
- For $y = \frac{-\sin(2\pi\sqrt{x})}{2\pi}$, TR gives Mirzakhani's recursion for Weil–Petersson volumes (of the moduli space of bordered hyperbolic surfaces), (Eynard–Orantin, '07, arXiv:0705.3600).
- TR on mirror curve of a toric CY3 computes its open Gromov–Witten theory (Bouchard–Klemm–Mariño–Pasquetti, '07, arXiv:0709.1453), (Fang–Liu–Zong, '16, arXiv:1604.07123).
- Statistical physics models on random maps: 1-hermitian matrix model, Ising model, Potts model, $O(\mathrm{n})$-loop model (Borot–Eynard, '09, arXiv:0910.5896), (Borot–Eynard–Orantin, '13, arXiv:1303.5808)...
- Semi-simple cohomological field theories and topological recursion (Dunin-Barkowski–Orantin–Shadrin–Spitz, '14, arXiv:1211.4021).
- Reconstruction of formal WKB expansions, integrability, isomonodromic systems (Borot–Eynard, '11, arXiv:1110.4936), (Eynard, '17, arXiv:1706.04938), (Eynard–G-F–Marchal–Orantin, '21, arXiv:2106.04339)...
- Conjecturally, for the $A$-polynomial of a knot as a spectral curve, TR computes the colored Jones polynomial of the knot (Borot–Eynard, '12, arXiv:1205.2261)).
- Equivalence with $W$-constraints (Kontsevich–Soibelman '17, ABCD of Andersen–Borot–Chekhov–Orantin '17, Borot–Bouchard–Chidambaram–Creutzig–Noshchenko '18 arXiv:1812.08738)

# Outline

# Witten's conjecture ⇝ Kontsevich's theorem



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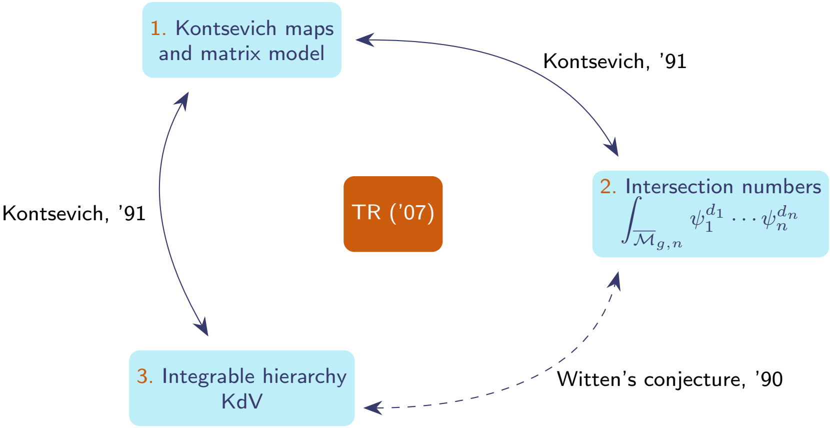

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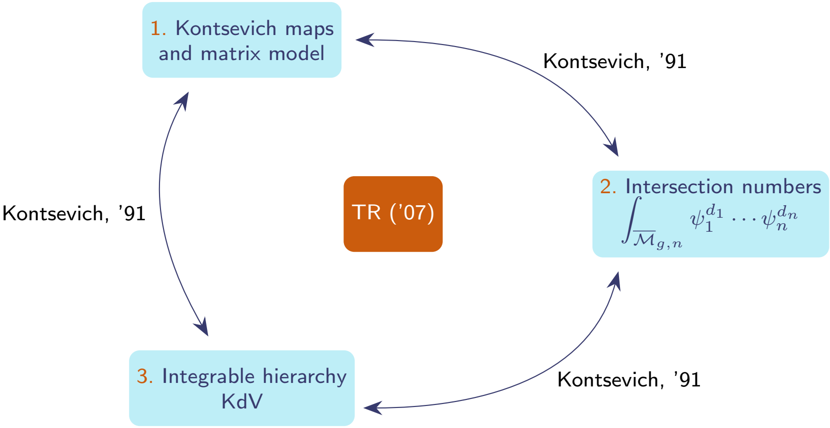

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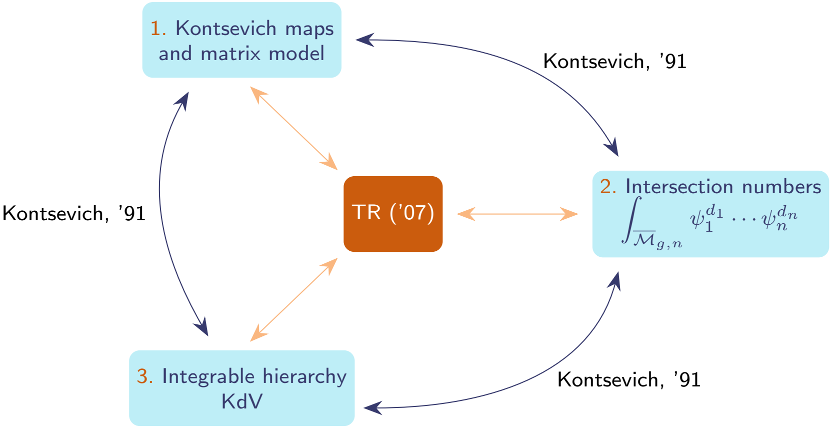


TR applied to the Airy curve $(x, y) = \left(\frac{z^2}{2}, z\right)$ produces

$$\omega_{g,n}(z_1, \ldots, z_n) = 2^{2-2g-n} \sum_{\sum_i d_i = 3g-3+n} \left( \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \right) \prod_{i=1}^{n} \frac{(2d_i+1)!! dz_i}{z_i^{2d_i+2}}.$$

# Airy differential equation

- Airy function $\mathrm{Ai}(\lambda) \rightsquigarrow \left(\frac{d^2}{d\lambda^2} - \lambda\right)\mathrm{Ai}(\lambda) = 0$. Asymptotic expansion as $\lambda \to \infty$ (g.s. of intersection numbers): $\log \mathrm{Ai}(\lambda) - S_0(\lambda) - S_1(\lambda) = \sum_{m=2}^{\infty} S_m(\lambda)$, where $S_0(\lambda) \coloneqq -\frac{2}{3}\lambda^{\frac{3}{2}}$, $S_1(\lambda) \coloneqq -\frac{1}{4}\log\lambda - \log(2\sqrt{\pi})$ and $\forall m \geq 2$

$$S_m(\lambda) \coloneqq \frac{\lambda^{-\frac{3}{2}(m-1)}}{2^{m-1}} \sum_{\substack{h \geq 0\,,\, n > 0 \\ 2h-2+n = m-1}} \frac{(-1)^n}{n!} \sum_{\mathbf{d} \in \mathbb{N}^n} \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{h,n} \prod_{i=1}^{n} (2d_i - 1)!!\,.$$

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- Formal parameter $\hbar$ to keep track of the Euler characteristics of the surfaces enumerated $\rightsquigarrow \psi^{\mathsf{Kont}}(\lambda, \hbar) := \mathrm{Ai}(\hbar^{-\frac{2}{3}}\lambda)$ satisfies

$$\left(\hbar^2 \frac{d^2}{d\lambda^2} - \lambda\right) \psi^{\mathsf{Kont}}(\lambda, \hbar) = 0$$

and admits an asymptotic expansion

$$\log \psi^{\mathsf{Kont}}(\lambda, \hbar) - \hbar^{-1} S_0(\lambda) - S_1(\lambda) = \sum_{m=2}^{\infty} \hbar^{m-1} S_m(\lambda).$$

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- TR on the Airy spectral curve $y^2 - x = 0$ computes $Z^{\mathsf{Kont}}(\hbar,\mathbf{t})$ and $\psi^{\mathsf{Kont}}(\lambda,\hbar)$, and allows to construct the *quantum curve* $\left(\hbar^2\frac{d^2}{d\lambda^2} - \lambda\right)\psi^{\mathsf{Kont}}(\lambda,\hbar) = 0$.

**General phenomenon?**

# Outline

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# Presentation of the quantum curve conjecture

$P \in \mathbb{C}[x,y]$ and $\Sigma = \{(x,y) \in \mathbb{C}^2 \mid P(x,y) = 0\}$ plane curve of genus $\hat{g}$.

A quantization of $\Sigma$ is a differential operator $\widehat{P}$ of the form

$$\widehat{P}(\widehat{x},\widehat{y};\hbar) = P_0(\widehat{x},\widehat{y}) + O(\hbar),$$

where $\widehat{x} = x\cdot$ , $\widehat{y} = \hbar \frac{d}{dx}$, such that $P_0(x,y) = P(x,y)$.

- The operators $\widehat{x}$ and $\widehat{y}$ satisfy $[\widehat{y},\widehat{x}] = \hbar$.
- $\widehat{P}(\widehat{x},\widehat{y})\psi(x,\hbar) = 0$. Schrödinger equation: $\left(\hbar^2 \frac{d^2}{dx^2} - \widehat{R}(\widehat{x},\hbar)\right)\psi(x,\hbar) = 0$.

WKB asymptotic expansion $\rightsquigarrow \log\psi(x,\hbar) = \sum_{k \geq -1} \hbar^k S_k(x) \in \hbar^{-1}\mathbb{C}[[\hbar]]$.

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**Conjecture**

*Both $\widehat{P}$ and $\psi$ can be constructed from $\Sigma$ using topological recursion.*

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*Both $\widehat{P}$ and $\psi$ can be constructed from $\Sigma$ using topological recursion.*

Subtlety: We want $\widehat{P}$ to have a controlled pole structure, more precisely, to have the same pole structure as $P$.

# Outline

# History and literature

- Proved for many particular cases $\rightsquigarrow$ genus $\hat{g} = 0$ spectral curves.
- Bouchard–Eynard '17 $\rightsquigarrow$ spectral curves whose Newton polygon has $N_I := \#\{\text{interior points}\} = 0$ (Fact: $\hat{g} \leq N_I$).

# History and literature

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- Mariño–Eynard '08 $\rightsquigarrow$ Holomorphic, modular and background independent, non-perturbative partition functions.
- Borot–Eynard '12 $\rightsquigarrow$ Only non-perturbative wave functions can obey "good" quantum curves (for $\hat{g} > 0$).
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- Chidambaram–Bouchard–Dauphinee '18 $\rightsquigarrow$ $\hat{g} = 1$, but bad properties (infinitely many $\hbar$ corrections with poles at ramification points, not even functions of $x$)!
- Iwaki–Marchal–Saenz '18, Marchal–Orantin '19 (reversed approach) $\rightsquigarrow$ Lax pairs associated with $\hbar$-dependent Painlevé equations and any $\hbar\partial_x\Psi(x,\hbar) = \mathcal{L}(x,\hbar)\Psi(x,\hbar)$, with $\mathcal{L}(x,\hbar) \in \mathfrak{sl}_2(\mathbb{C})$, satisfy the topological type property from Bergère–Borot–Eynard '15 ($\hat{g} = 0$).
- Iwaki–Saenz '16, Iwaki '19 $\rightsquigarrow$ Painlevé I and elliptic curves ($\hat{g} = 1$).

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- Iwaki–Marchal–Saenz '18, Marchal–Orantin '19 (reversed approach) $\rightsquigarrow$ Lax pairs associated with $\hbar$-dependent Painlevé equations and any $\hbar\partial_x\Psi(x,\hbar) = \mathcal{L}(x,\hbar)\Psi(x,\hbar)$, with $\mathcal{L}(x,\hbar) \in \mathfrak{sl}_2(\mathbb{C})$, satisfy the topological type property from Bergère–Borot–Eynard '15 ($\hat{g} = 0$).
- Iwaki–Saenz '16, Iwaki '19 $\rightsquigarrow$ Painlevé I and elliptic curves ($\hat{g} = 1$).
- Marchal–Orantin '19, Eynard–GF '19 $\rightsquigarrow$ Hyperelliptic (any $\hat{g}$).

# History and literature

- Proved for many particular cases ⇝ genus $\hat{g} = 0$ spectral curves.
- Bouchard–Eynard '17 ⇝ spectral curves whose Newton polygon has $N_I := \#\{\text{interior points}\} = 0$ (Fact: $\hat{g} \leq N_I$).
- Mariño–Eynard '08 ⇝ Holomorphic, modular and background independent, non-perturbative partition functions.
- Borot–Eynard '12 ⇝ Only non-perturbative wave functions can obey "good" quantum curves (for $\hat{g} > 0$).
- Eynard '17 ⇝ General idea to construct integrable systems and their $\tau$-functions from the geometry of the spectral curve.
- Chidambaram–Bouchard–Dauphinee '18 ⇝ $\hat{g} = 1$, but bad properties (infinitely many $\hbar$ corrections with poles at ramification points, not even functions of $x$)!
- Iwaki–Marchal–Saenz '18, Marchal–Orantin '19 (reversed approach) ⇝ Lax pairs associated with $\hbar$-dependent Painlevé equations and any $\hbar \partial_x \Psi(x, \hbar) = \mathcal{L}(x, \hbar) \Psi(x, \hbar)$, with $\mathcal{L}(x, \hbar) \in \mathfrak{sl}_2(\mathbb{C})$, satisfy the topological type property from Bergère–Borot–Eynard '15 ($\hat{g} = 0$).
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- Eynard–GF–Marchal–Orantin '21 ⇝ any algebraic curve with simple ramifications.

# Beyond Airy: some meaningful generalisations

- $y^2 = x \rightsquigarrow$ Witten (conj) '90, Kontsevich '91, Airy, KW KdV tau function

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}$$

$$\left(\hbar^2 \frac{d^2}{dx^2} - x\right)\psi(z,\hbar) = 0$$

- $y^2 x = 1 \rightsquigarrow$ Norbury (conj) '17, Chidambaram, Giacchetto, G-F, '22, Bessel, BGW KdV tau function

$$\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \psi_1^{d_1} \cdots \psi_n^{d_n}$$

$$\left(\hbar^2 \frac{d}{dx} x \frac{d}{dx} - 1\right)\psi(z,\hbar) = 0$$

- $y^r = x \rightsquigarrow$ Witten '93, Faber–Shadrin–Zvonkine, '10, $r$Airy, $r$KdV

$$\int_{\overline{\mathcal{M}}_{g,n}} W^r_{g,n}(a_1,\dots,a_n)\psi_1^{d_1} \cdots \psi_n^{d_n}$$

$$\left(\hbar^r \frac{d^r}{dx^r} - x\right)\psi(z,\hbar) = 0$$

- $y^2 = x^3 + tx + V \rightsquigarrow$ **Painlevé I**, elliptic curve ($\hat{g} = 1$)

$$\int_{\overline{\mathcal{M}}_{g,n+m}} \psi_{n+1}^2 \cdots \psi_{n+m}^2 \psi_1^{d_1} \cdots \psi_n^{d_n}$$

$$\left(\hbar^2 \frac{d^2}{dx^2} - \left(x^3 + tx + V + \frac{\partial}{\partial t}\right)\right)\psi = 0$$

# Outline

# Spectral curves

$N$ distinct points $\Lambda_1, \ldots, \Lambda_N \in \mathbb{P}^1 \setminus \{\infty\}$. Let $\mathcal{H}_d(\Lambda_1, \ldots, \Lambda_N, \infty)$ be the Hurwitz space of degree $d$ ramified coverings $x\colon \Sigma \to \mathbb{P}^1$, where $\Sigma$ is the Riemann surface:

$$\Sigma := \overline{\{(\lambda, y) \mid P(\lambda, y) = 0\}}$$

of genus $\hat{g}$, where $x(\lambda, y) := \lambda$ and

$$P(\lambda, y) = \sum_{l=0}^{d} (-1)^l y^{d-l} P_l(\lambda), \quad P_0(\lambda) = 1,$$

$P_l$ being a rational function with possible poles at $\lambda \in \mathcal{P} := \{\Lambda_i\}_{i=1}^N \bigcup \{\infty\}$.

**Classical spectral curve:** $\rightsquigarrow (\Sigma, x)$.

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**Definition (Admissible classical spectral curves)**

A classical spectral curve $(\Sigma, x)$ is *admissible* if:

- $P(\lambda, y) = 0$ is an irreducible algebraic curve;
- $a \in \mathrm{Ram}(x)$ are simple, i.e. $dx$ has only a simple zero at $a \in \mathcal{R}$;
- $\forall a \in \mathcal{R}$, $dy(a) \neq 0$;

# Torelli marking and filling fractions

Fix a symplectic basis $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^{\hat{g}}$ of $H_1(\Sigma, \mathbb{Z})$ and a Lagrangian $\mathcal{L}$ associated to the $\mathcal{A}$-cycles.

**Remark**

*Choice of Torelli marking can be thought of as a choice of polarisation.*

# Torelli marking and filling fractions

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**Remark**

*Choice of Torelli marking can be thought of as a choice of polarisation.*

Let $\left((\Sigma, x), (\mathcal{A}_i, \mathcal{B}_i)_{i=1}^{\hat{g}}\right)$ be some admissible initial data. We define the tuple $(\epsilon_i)_{i=1}^{\hat{g}}$ of *filling fractions* by

$$\forall\, i \in [\![1, \hat{g}]\!]\,, \quad \epsilon_i := \frac{1}{2\pi i} \oint_{\mathcal{A}_i} y dx.$$

$\omega_{0,1}(z) = y(z)dx(z)$, $\omega_{0,2}(z_1, z_2) = B^{\mathcal{L}}(z_1, z_2) \Rightarrow$

$$\frac{\partial}{\partial \epsilon_i} \omega_{h,n}(z_1, \ldots, z_n) = \oint_{z \in \mathcal{B}_i} \omega_{h,n+1}(z, z_1, \ldots, z_n), \forall i \in [\![1, \hat{g}]\!].$$

# Outline

1. Topological recursion and its ramifications
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   - One of our playgrounds: Witten–Kontsevich and Airy
2. The quantisation problem
   - Quantum curve problem
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   - Perturbative wave functions
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# Loop equations

$Q^{(l)}_{h,n+1}(\lambda;\mathbf{z})$ symmetric algebraic combinations of the $\omega_{g,n}$s taken at all preimages $x^{-1}(\lambda)$.

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**Theorem (Loop equations)**

*The function* $\lambda \mapsto \frac{Q^{(l)}_{h,n+1}(\lambda;\mathbf{z})}{(d\lambda)^l}$ *has no poles at* $\lambda \in x(\mathcal{R})$, $\forall \mathbf{z} \in (\Sigma \setminus \mathcal{R})^n$.

**Linear:**

- $$Q^{(1)}_{h,n+1}(\lambda;\mathbf{z}) = \sum_{z\in x^{-1}(\lambda)} \omega_{h,n+1}(z,\mathbf{z}) = \delta_{n,0}\delta_{h,0}P_1(\lambda)d\lambda + \delta_{n,1}\delta_{h,0}\frac{d\lambda\, dx(z_1)}{(\lambda - x(z_1))^2}.$$

# Outline

# Perturbative wave function over a divisor

$D = \sum_{i=1}^{s} \alpha_i [p_i]$ a generic divisor (of degree$= \sum_i \alpha_i = 0$) on $\widetilde{\Sigma_{\mathcal{P}}}$, $\Sigma_{\mathcal{P}} \coloneqq \Sigma \setminus x^{-1}(\mathcal{P})$.

*Perturbative wave function* $\psi(D, \hbar)$ associated to $D$:

$$\exp\left(\sum_{h\geq 0}\sum_{n\geq 0} \frac{\hbar^{2h-2+n}}{n!} \int_D \cdots \int_D \left(\omega_{h,n}(z_1,\dots,z_n) - \delta_{h,0}\delta_{n,2}\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}\right)\right).$$

$$e^{-\hbar^{-2}\omega_{0,0}} e^{-\hbar^{-1}\int_D \omega_{0,1}} \psi(D,\hbar) \in \mathbb{C}[[\hbar]].$$

$\psi(D = [z] - [p_2], \hbar)$ has an essential singularity at $p_2 \to \infty^{(\alpha)}$ $\rightsquigarrow$ Need to regularise $\psi$ and KZ equations.

*Perturbative partition function* $Z(\hbar) = \psi(D = \emptyset, \hbar)$:

$$Z(\hbar) \coloneqq \exp\left(\sum_{h\geq 0} \hbar^{2h-2}\omega_{h,0}\right), \text{ with } e^{-\hbar^{-2}\omega_{0,0}} Z(\hbar) \in \mathbb{C}[[\hbar]].$$

**Remark**

*Wave functions $\rightsquigarrow$ solutions to a differential equation; the partition function $\rightsquigarrow$ role of tau function from the point of view of isomonodromic or integrable systems.*

# Outline

# KZ equations for $d = 2 \rightsquigarrow$ system of PDEs

Loop equations can be combined into a g. s. to form a system of $d \times s$ “differential equations” satisfied by the wave functions. Case $d = 2$:

**Theorem (Eynard–GF,’19)**

*For* $k = 1, 2$,

$$\hbar^2 \left( \frac{d^2}{dx_k^2} + \sum_{i \neq k} \frac{\frac{d}{dx_k} - \frac{d}{dx_i}}{x_k - x_i} \right) \psi = \left( R(x_k) + \mathcal{L}(x_k) \right) \psi.$$

$\zeta_\infty \in x^{-1}(\infty)$ and $\zeta_l \in x^{-1}(\Lambda_l)$ poles of $\omega_{0,1}$ of orders $m_\infty$ and $m_l$, $l = 1, \ldots, N$, respectively. Let $d_\infty := \mathrm{ord}_{\zeta_\infty}(x)$. Operator $\mathcal{L}(x) = \mathcal{L}_\infty(x) + \mathcal{L}_\Lambda(x)$:

$$\mathcal{L}_\infty(x) = \sum_{j=1-2d_\infty}^{m_\infty} t_{\zeta_\infty, j} \sum_{k=0}^{\frac{1-j}{d_\infty} - 2} x^k \left( -\frac{j}{d_\infty} - k - 2 \right) \frac{\partial}{\partial t_{\zeta_\infty, j + d_\infty(k+2)}},$$

$$\mathcal{L}_\Lambda(x) = \sum_{l=1}^{N} \left( \frac{1}{x - \lambda_l} \frac{\partial}{\partial \lambda_l} + \sum_{j=1}^{m_l - 1} t_{\zeta_l, j} \sum_{k=1}^{j} (x - \lambda_l)^{-(k+1)} (j + 1 - k) \frac{\partial}{\partial t_{\zeta_l, j+1-k}} \right).$$

**Example**

In the Airy case, $y^2 = x$, we have only one pole, at $\zeta_i = \infty$, of degree $m_i = 3$, with $d_i = -2$. The sum is empty and $\mathcal{L}(x) = 0$.

## Airy and elliptic cases for two-point divisors

Divisor $D = [z_1] - [z_2]$:

- **PDEs for Airy curve**: $y^2 = x$. We had $\mathcal{L}(x) = 0$.

$$\begin{cases} \hbar^2 \left( \frac{d^2}{dx_1^2} + \frac{\frac{d}{dx_1} - \frac{d}{dx_2}}{x_1 - x_2} \right) \psi & = x_1 \psi, \\ \hbar^2 \left( \frac{d^2}{dx_2^2} + \frac{\frac{d}{dx_1} - \frac{d}{dx_2}}{x_1 - x_2} \right) \psi & = x_2 \psi. \end{cases}$$

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More generally, admissible curves considered in Bouchard–Eynard, '17 (empty Newton polygon) are those for which $\mathcal{L}(x) = 0$.

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More generally, admissible curves considered in Bouchard–Eynard, '17 (empty Newton polygon) are those for which $\mathcal{L}(x) = 0$.

- **PDEs for elliptic curve**: $R(x(z)) = y(z)^2 = x^3 + tx + V$, with

$$-V = \int_{\mathcal{B}_{\infty,1}} \omega_{0,1} = \frac{\partial}{\partial t_{\infty,1}} \omega_{0,0} = -\frac{\partial}{\partial t} \omega_{0,0}$$

$\Rightarrow R(x(z)) = x^3 + tx + \frac{\partial}{\partial t} \omega_{0,0}$.
We have $\mathcal{L}(x) = \frac{\partial}{\partial t}$.

$$\left( \hbar^2 \frac{d^2}{dx_k^2} + \hbar^2 \frac{\frac{d}{dx_1} - \frac{d}{dx_2}}{x_1 - x_2} \right) \psi = (x_k^3 + tx_k + V + \frac{\partial}{\partial t}) \psi,$$

for $k = 1, 2$.

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# Monodromies of the perturbative wave function $\rightsquigarrow$ bad monodromies

Problem for genus $\hat{g} > 0$: $\int_o^z \cdots \int_o^z \omega_{g,n}$ are not invariant after $z$ goes around a cycle. Very bad monodromies when $z$ goes around a $\mathcal{B}_i$ (first type cycle).

**Lemma**

$$\forall\, j \in [\![1, \hat{g}]\!] \;:\; \psi([z + \mathcal{A}_j] - [\infty^{(\alpha)}], \hbar) = e^{\frac{2\pi i \epsilon_j}{\hbar}} \psi([z] - [\infty^{(\alpha)}], \hbar),$$

$$\psi(D + \mathcal{B}_j, \hbar) = \exp\left( \sum_{(h,n,m)\in\mathbb{N}^3} \frac{\hbar^{2h-2+n+m}}{n!m!} \overbrace{\int_D \cdots \int_D}^{n} \overbrace{\int_{\mathcal{B}_j} \cdots \int_{\mathcal{B}_j}}^{m} \omega_{h,n+m} \right).$$

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Since the $\mathcal{B}_j$ period of $\omega_{h,n+1}$ is equal to the variation of $\omega_{h,n}$ wrt $\epsilon_j := \oint_{\mathcal{A}_j} \omega_{0,1}$,

$$\psi(D + \mathcal{B}_j, \hbar) = \exp\left( \sum_{(h,n)\in\mathbb{N}^2} \frac{\hbar^{2h-2+n}}{n!} \overbrace{\int_D \cdots \int_D}^{n} \sum_{m\geq 0} \frac{1}{m!} \left(\hbar \frac{\partial}{\partial \epsilon_j}\right)^m \omega_{h,n} \right) \Rightarrow$$

$$\psi([z + \mathcal{B}_j] - [\infty^{(\alpha)}], \hbar) = e^{\hbar \frac{\partial}{\partial \epsilon_j}} \psi([z] - [\infty^{(\alpha)}], \hbar) = \psi([z] - [\infty^{(\alpha)}], \hbar, \epsilon_j \to \epsilon_j + \hbar).$$

# Outline

# Summing over the lattice

**Remark**

*Our KZ equations do not depend on $z \in \Sigma$ but only on its image $x(z)$ $\Rightarrow$*
*For any finite family of $c_\gamma$, the following sum satisfies the same KZ equations*

$$\psi_l([z] - [\infty^{(\alpha)}], \hbar, \{c_\gamma\}) := \sum_{\gamma \in \pi_1(\Sigma \setminus x^{-1}(\mathcal{P}))} c_\gamma \; \psi_l([z] + \gamma - [\infty^{(\alpha)}], \hbar).$$

Goal: Build solutions to the same KZ equations but with better monodromies along the $\mathcal{B}_i$-cycles.

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Goal: Build solutions to the same KZ equations but with better monodromies along the $\mathcal{B}_i$-cycles.

Strategy: Sum over $\gamma = \sum_{i=1}^{g} n_i \mathcal{B}_i$, i.e. $\epsilon_i \to \epsilon_i + \hbar$. Formally $\rightsquigarrow$ discrete Fourier transform of the perturbative wave function:

$$\psi_l^{\infty^{(\alpha)}}(z, \hbar; \boldsymbol{\epsilon}, \boldsymbol{\rho}) := \sum_{\mathbf{n} \in \mathbb{Z}^g} e^{\frac{2\pi i}{\hbar} \sum_{j=1}^{g} \rho_j n_j} \; \psi_l([z] - [\infty^{(\alpha)}], \hbar, \epsilon + \hbar \mathbf{n}).$$

# Trans-series with special ordering

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**Remark (Limitations)**

- *Filling fraction* $\boldsymbol{\epsilon} = (\epsilon_1,\dots,\epsilon_g) \rightsquigarrow$ *not a global coordinate on the space of classical spectral curves with fixed spectral times (only a local coordinate).*

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We need a special ordering of the trans-monomials:

$$\sum_{r\geq 0}\sum_{\mathbf{n}\in\mathbb{Z}^{\hat{g}}} F_{\mathbf{n},r}\hbar^r e^{\frac{1}{\hbar}\sum_{j=1}^{\hat{g}} n_j v_j}.$$

The partial sums $\sum_{\mathbf{n}\in\mathbb{Z}^{\hat{g}}} F_{\mathbf{n},r} e^{\frac{1}{\hbar}\sum_{j=1}^{\hat{g}} n_j v_j}$ will give rise to theta functions.

Equalities: coefficient by coefficient in the trans-monomials.

# Non-perturbative wave functions

**Riemann matrix of periods of $\Sigma$**: $\tau_{i,j} = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{B}_i} \int_{\mathcal{B}_j} \omega_{0,2}$, $\forall (i,j) \in [\![1,\hat{g}]\!]^2$.

Riemann theta function (analytic function of $\mathbf{v} \in \mathbb{C}^{\hat{g}}$) and its derivatives:

$$\Theta^{(i_1,\dots,i_k)}(\mathbf{v},\tau) = \sum_{(n_1,\dots,n_{\hat{g}})\in\mathbb{Z}^{\hat{g}}} e^{2\pi \mathrm{i} \sum_{i=1}^{\hat{g}} n_i v_i} e^{\pi \mathrm{i} \sum_{(i,j)\in[\![1,\hat{g}]\!]^2} n_i \tau_{i,j} n_j} \prod_{j=1}^{k} n_{i_j}.$$

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$D = [z] - [\infty^{(\alpha)}] \rightsquigarrow$ non-perturbative wave function

$$\psi_{\mathrm{NP}}(D;\hbar,\boldsymbol{\rho}) := e^{\hbar^{-2}\omega_{0,0}+\omega_{1,0}} e^{\hbar^{-1}\int_D \omega_{0,1}} \frac{1}{E(D)} \sum_{r=0}^{\infty} \hbar^r G^{(r)}(D;\boldsymbol{\rho}),$$

where $E$ is the prime form on $\Sigma$,

$$G^{(r)}(D;\boldsymbol{\rho}) := \sum_{k=0}^{3r} \sum_{i_1,\dots,i_k\in[\![1,\hat{g}]\!]^k} \Theta^{(i_1,\dots,i_k)}(\mathbf{v},\tau) G^{(r)}_{(i_1,\dots,i_k)}(D)$$

and where $v_j := \frac{\rho_j+\varphi_j}{\hbar} + \mu_j^{(\alpha)}(z)$, $\mathbf{v} = (v_1,\dots,v_{\hat{g}})$, with

$$\varphi_j := \frac{1}{2\pi i} \oint_{\mathcal{B}_j} \omega_{0,1} \quad \text{and} \quad \mu_j^{(\alpha)}(z) := \frac{1}{2\pi i} \int_D \oint_{\mathcal{B}_j} \omega_{0,2}.$$

# Outline

# Same KZ equations and good monodromies

- Non-perturbative wave functions satisfy the same KZ equations as their perturbative partners.

$$\hbar \frac{d\psi_{l,\mathrm{NP}}^{\infty^{(\alpha)}}(z,\hbar,\boldsymbol{\rho})}{dx(z)} + \psi_{l+1,\mathrm{NP}}^{\infty^{(\alpha)}}(z,\hbar,\boldsymbol{\rho}) =$$

$$\sum_{P\in\mathcal{P}} \sum_{k\in S_P^{(l+1)}} \xi_P^{-k}(x(z))\mathrm{ev.}\left[\widetilde{\mathcal{L}}_{P,k,l}\,\psi_{0,\mathrm{NP}}^{\infty^{(\alpha)},\,\mathrm{symbol}}(z,\hbar,\boldsymbol{\rho})\right].$$

- Non-perturbative wave functions $\rightsquigarrow$ simple monodromy properties.

For $j\in[\![1,\hat{g}]\!]$, we have

$$\psi_{l,\mathrm{NP}}^{\infty^{(\alpha)}}(z+\mathcal{A}_j,\hbar,\boldsymbol{\rho}) = e^{\frac{2\pi i\epsilon_j}{\hbar}}\,\psi_{l,\mathrm{NP}}^{\infty^{(\alpha)}}(z,\hbar,\boldsymbol{\rho}),$$

$$\psi_{l,\mathrm{NP}}^{\infty^{(\alpha)}}(z+\mathcal{B}_j,\hbar,\boldsymbol{\rho}) = e^{-\frac{2\pi i\rho_j}{\hbar}}\,\psi_{l,\mathrm{NP}}^{\infty^{(\alpha)}}(z,\hbar,\boldsymbol{\rho})$$

and $\forall\ p\in x^{-1}(\mathcal{P})$

$$\psi_{l,\mathrm{NP}}^{\infty^{(\alpha)}}(z+\mathcal{C}_p,\hbar,\boldsymbol{\rho}) = (-1)^{\delta_{p,\infty^{(\alpha)}}}\ e^{\frac{2\pi i t_{p,0}}{\hbar}}\,\psi_{l,\mathrm{NP}}^{\infty^{(\alpha)}}(z,\hbar,\boldsymbol{\rho}).$$

# Outline

# Lax systems

For $l \geq 0$, we define

$$\psi^{\infty(\alpha)}_{l,\mathrm{NP}}(z,\hbar,\boldsymbol{\rho}) := \mathrm{ev}. \sum_{\beta \underset{l}{\subseteq} \left(x^{-1}(x(z))\setminus\{z\}\right)} \frac{1}{l!} \left( \prod_{j=1}^{l} \mathcal{I}_{\mathcal{C}_{\beta_j,1}} \right) \psi^{\mathrm{symbol}}_{\mathrm{NP}}(D;\hbar,\boldsymbol{\rho}).$$

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We use them to define a $d \times d$ matrix

$$\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar,\boldsymbol{\rho}) := \left[ \psi^{\infty(\alpha)}_{l-1,\mathrm{NP}}(z^{(\beta)}(\lambda),\hbar,\boldsymbol{\rho}) \right]_{1 \leq l,\beta \leq d},$$

where $z^{(\beta)}(\lambda)$ denotes the $\beta^{\text{th}}$ preimage by $x$ of $\lambda$.

# Lax systems

**Theorem (ODE and Lax system)**

*Let* $\hat{L}(\lambda,\hbar) := -\widehat{P}(\lambda) + \hbar \sum_{P\in\mathcal{P}} \sum_{k\in\mathbb{N}} \xi_P^{-k}(\lambda)\widehat{\Delta}_{P,k}(\lambda,\hbar)$. *Then,*

$$\hbar \frac{d\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar)}{d\lambda} = \hat{L}(\lambda,\hbar)\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar),$$

*where*

$$\widehat{P}(\lambda) := \begin{bmatrix} -P_1(\lambda) & 1 & 0 & \dots & 0 \\ -P_2(\lambda) & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -P_{d-1}(\lambda) & 0 & 0 & \dots & 1 \\ -P_d(\lambda) & 0 & 0 & \dots & 0 \end{bmatrix}$$

*For any* $P \in \mathcal{P}$, $k \in \mathbb{N}$, $l \in [\![0, d-1]\!]$, *one has the auxiliary systems*

$$\hbar^{-1}\mathrm{ev}.\mathcal{L}_{P,k,l}\widehat{\Psi}_{\mathrm{NP}}^{\mathrm{symbol}}(\lambda,\hbar) = \widehat{A}_{P,k,l}(\lambda,\hbar)\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar),$$

*where* $\hat{L}(\lambda,\hbar)$ *and* $\widehat{A}_{P,k,l}(\lambda,\hbar)$ *are* $\hbar$*-trans-series functions that are rational functions of* $\lambda$*, with no poles at critical values* $\lambda \in x(\mathcal{R})$.

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- (1) $\rightsquigarrow$ linear differential system of size $d \times d$ whose formal fundamental solution can be computed by TR, with poles at the poles of the leading WKB term...
- $\hat{L}(\lambda,\hbar)$ has poles only at $\lambda \in \mathcal{P}$ and at zeros of the Wronskian $\det \widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar)$, *apparent singularities* of the system (can be computed thanks to the KZ eqns).

# Lax systems

**Theorem (ODE and Lax system)**

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$$\hbar \frac{d\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar)}{d\lambda} = \hat{L}(\lambda,\hbar)\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar), \tag{2}$$

*where*

$$\widehat{P}(\lambda) := \begin{bmatrix} -P_1(\lambda) & 1 & 0 & \dots & 0 \\ -P_2(\lambda) & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -P_{d-1}(\lambda) & 0 & 0 & \dots & 1 \\ -P_d(\lambda) & 0 & 0 & \dots & 0 \end{bmatrix}$$

*For any* $P \in \mathcal{P}$, $k \in \mathbb{N}$, $l \in [\![0, d-1]\!]$, *one has the auxiliary systems*

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- Most technical proof $\rightsquigarrow$ by induction on the order of the transseries.
- Proof uses admissibility conditions (distinct critical values, smooth simple ramification points) $\rightsquigarrow$ should adapt without them but involving more technical computations.

# 4 different interesting gauges and examples

None of the gauge transformations modify the first line of the wave functions matrix (used to define the quantum curve).

- **Gauge** $\widehat{\Psi}$: Gauge coming from KZ equations which provides compatible auxiliary systems $\left(\mathcal{L}_{P,k,l}\right)_{P\in\mathcal{P},l\in[\![0,d-1]\!],k\in S_P^{(l+1)}}$.
- **Gauge** $\widetilde{\Psi}$ ($\hbar^0$ gauge transformation from $\widehat{\Psi}$): Leading order in $\hbar$ of $\widetilde{L}$ is companion-like $\rightsquigarrow$ the classical spectral curve directly recovered from last line.
- **Gauge** $\Psi$: Lax matrix $L$ is companion-like at all orders in $\hbar$ $\rightsquigarrow$ both the quantum and classical curves directly read from the last line of $L$ and its $\hbar\to 0$ limit. Natural framework for Darboux coordinates and isomonodromic deformations.
- **Gauge** $\check{\Psi}$: Lax matrix $\check{L}$ has no apparent singularities $\rightsquigarrow$ $\check{L}(\lambda,\hbar)d\lambda$ as an $\hbar$-familly of Higgs fields giving rise to a flow in the corresponding Hitchin system.

**Example**

- Reconstruction via TR of a 2-parameter family of formal transseries solutions to Painlevé 2 and quantization. Classical spectral curve: $y^2-P_1(\lambda)y+P_2(\lambda)=0$, where $P_1(\lambda)=P_{\infty,2}^{(1)}\lambda^2+P_{\infty,1}^{(1)}\lambda+P_{\infty,0}^{(1)}$ and $P_2(\lambda)=P_{\infty,4}^{(2)}\lambda^4+P_{\infty,3}^{(2)}\lambda^3+P_{\infty,2}^{(2)}\lambda^2+P_{\infty,1}^{(2)}\lambda+P_{\infty,0}^{(2)}$.
- Quantisation of a degree 3, genus 1 classical spectral curve with a single singularity at infinity: $y^3-(P_{\infty,1}^{(1)}\lambda+P_{\infty,0}^{(1)})y^2+(P_{\infty,2}^{(2)}\lambda^2+P_{\infty,1}^{(2)}\lambda+P_{\infty,0}^{(2)})y-P_{\infty,3}^{(3)}\lambda^3-P_{\infty,2}^{(3)}\lambda^2-P_{\infty,1}^{(3)}\lambda-P_{\infty,0}^{(3)}=0$.

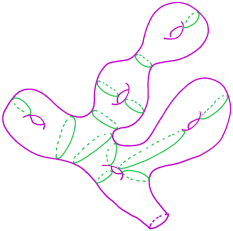
# Outline

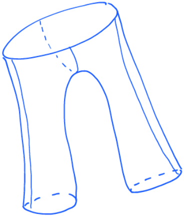
1. Topological recursion and its ramifications
   - Introduction, connections, examples
   - One of our playgrounds: Witten–Kontsevich and Airy
2. The quantisation problem
   - Quantum curve problem
   - Origins, context and examples
   - (Classical) spectral curves
3. From loop equations to KZ equations
   - Perturbative wave functions
   - KZ-like equations
   - Bad monodromies
4. Non-perturbative wave functions and Lax system
   - Good monodromies
   - Lax systems
5. Present and future
6. 3 applications: critical exponents, non-perturbative JT, free probability

# Some of my questions

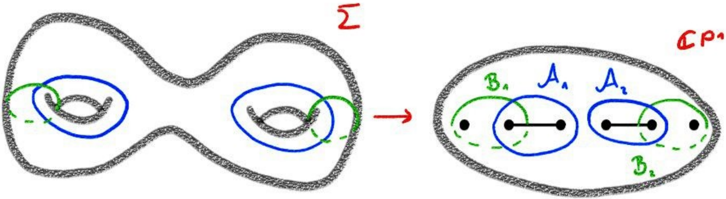
- Explore the connection with summability, exact WKB, Stokes phenomenon and resurgence. **Conjecture:** There exist values of $\varepsilon$ and $\hbar$ making the transseries involved summable.
- **Conjecture:** The non-perturbative partition function is a tau function.
- How does the connection built as $d - \mathcal{L}(x, \hbar)dx/\hbar$ depend on the choice of cycles $(\mathcal{A}_i, \mathcal{B}_i)$?
- Remove resurgence assumption from our proof of large genus asymptotics of Weil–Petersson volumes.
- Interesting enumerative geometry in higher genus TR problems?
- Extend TR beyond orientable surfaces: Klein surfaces, non-orientable enumerative geometry and real moduli space.
- Master $x - y$ swap transformation.

**Your questions?**





Merci beaucoup pour votre attention !



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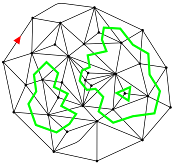
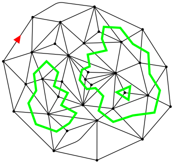
# Large maps with small and large boundaries, and with loops



"The geometry of large random maps is **universal**"

- $O(n)$ loop model ⇝ statistical ensemble of maps endowed with loop configurations.
- 2 new universality classes (depending continuously on n) ⇝ dense and dilute.

# Large maps with small and large boundaries, and with loops



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- $O(\mathrm{n})$ loop model ⇝ statistical ensemble of maps endowed with loop configurations.

- 2 new universality classes (depending continuously on n) ⇝ dense and dilute.

1. G. Borot, J. Bouttier et B. Duplantier ⇝ nesting properties $(0,1)$ and $(0,2)$.
2. Analysis of critical behavior of TR in the presence of large and small boundaries.
3. Nesting properties for arbitrary topologies.

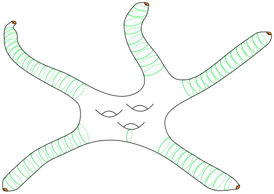
When $V \to \infty$:

- Typical configuration with small boundaries ⇝ probably incident to distinct arms (with $O(\ln V)$ separating loops).

[Borot–G-F arXiv:1609.02074]

Let $\mathfrak{d} = 1(-1)$ and $c = \frac{1}{1-b}(1)$ in the dense (dilute) phase, with $b(\mathrm{n}) \in (\frac{1}{2}, 0)$.
For $2g - 2 + k > 0$, when $u \to 1^-$, we have for g.s. of configurations with $k_S$ small boundaries

$$\mathrm{Conf}_k^{[g]}(x_1, \ldots, x_k) \stackrel{\cdot}{\sim} (1-u)^{c((2\mathrm{g}-2+k)(\mathfrak{d}\frac{b}{2}-1)-\frac{k}{2}+\frac{3}{4}k_S)}.$$

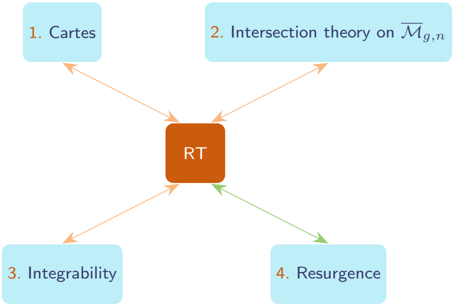


# Non perturbative TR and resurgence

Natural extension of perturbative expansions to transseries:

$$\underbrace{\sum_{g\geq 0} a_g^{(0)} \hbar^n}_{\text{perturbative expansion}} + \underbrace{e^{-A/\hbar} \sum_{g\geq 0} a_g^{(1)} \hbar^g}_{\text{1st instanton sector}} + \underbrace{O(e^{-2A/\hbar})}_{\text{higher instanton corrections}} .$$

behavior of $a_g^{(0)}$ when $g \to \infty$ $\leftrightarrow$ $a_0^{(1)}$ (+corrections)



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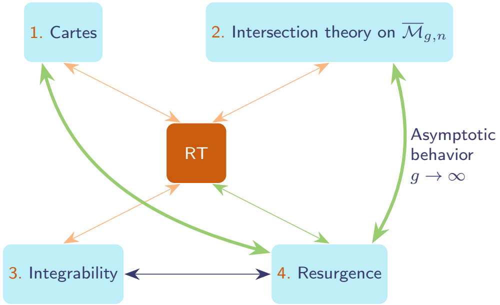

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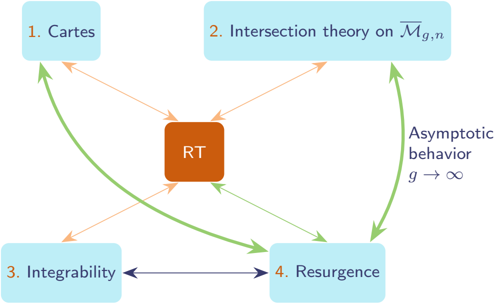


**Recent work:**

A [Eynard–G-F–Gregori–Lewański–Schiappa, '23 arXiv:2305.16940]: non-perturbative corrections to JT gravity via TR, geometric interpretation of instanton corrections and large genus asymptotics of Weil–Petersson volumes (assuming resurgence!).

B [Eynard–G-F–Giacchetto–Gregori–Lewański, '23 arXiv:2309.03143]: Large genus asymptotics of intersection numbers (with no assumptions!).

# A triple duality: symplectic, simple and free

Through monotone Hurwitz numbers

- **Free probability:**

  Moments $\varphi \quad \leftrightarrow \quad$ Free cumulants $\kappa$

  [Borot, Charbonnier, Leid, Shadrin, G-F, '21 arXiv:2112.12184]

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- **Combinatorics:**

  Maps $\quad \leftrightarrow \quad$ Fully simple maps

  [Borot, G-F, '17 arXiv:1710.07851]
  [Borot, Charbonnier, Do, G-F, '19 arXiv:1904.02267]

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- **Topological recursion (TR):**

  $$(x,y) \overset{\text{TR}}{\rightsquigarrow} \omega_{g,n} \quad \leftrightarrow \quad (\check{x},\check{y}) \overset{\text{TR}}{\rightsquigarrow} \check{\omega}_{g,n},$$

  with $\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}\check{x} \wedge \mathrm{d}\check{y}$ (symplectic transformation).

  [Alexandrov, Bychkov, Dunin-Barkowski, Kazarian, Shadrin, '21 arXiv:2212.00320]

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- **Quantum curves:** Harnad duality?