p-adic periods and the connectivity of the motivic Hopf algebra

I. Intro

p-adic analogs / char p analogs of:

1) the notion of period / $\mathbb{C}$

2) the connectivity property of the motivic Hopf algebra

Remark: $\forall$ X over $\mathbb{Q}$, alg var / motive

$\omega \in H^n_{dR}(X)$, $\gamma \in H_n^{Sing}(X)$

$\int_\gamma \omega \in \mathbb{C}$

Q.) Is there a p-adic version of this?

Can one get some elements in $\mathbb{Q}_p$ in a similar manner?

Remark: In ch. 0, $k \subset \mathbb{C}$.

$\rightsquigarrow$ motivic Hopf algebra $\mathcal{H}_{mot}(k)$.

$\uparrow$

dg - Hopf alg

$\mathrm{Spec}(\mathcal{H}_{mot}) = \mathcal{G}_{mot}$ acts on singular cohomology of motives / k in a universal way.

known Fact: $H_i(\mathcal{H}_{mot}) = 0 \quad i < 0.$

$$\underbrace{\mathcal{G}_{mot}^{cl} \subset \mathcal{G}_{mot}}_{\text{Nori's motivic Galois group}}$$

$\mathcal{G}_{mot}^{cl} = \mathrm{Spec}\, H_0(\mathcal{H}_{mot}).$

Q.) Is there a version of this in $\mathrm{ch} > 0$.

II) Construction of p-adic periods

X over $\mathbb{Q}$ a variety / motive.

$\omega \in H^n_{dR}(X)$ a de Rham coh. class

Q.) What would be the analog of topological chains?

A.) Suslin homology of X^{an} = the rigid analytic variety / $\mathbb{Q}_p$ associated to X.

Reminder: X a variety / k.

$(Z \longrightarrow X)$

$$\mathrm{Cor}(\Delta^{\bullet}_{alg}, X) \quad \Big| \quad \Sigma\Big(\Delta^{n}_{alg} \Big)$$

$$\Delta^n_{alg} = \mathrm{Spec}\left(k[t_0, \dots, t_n]/t_0 + \dots + t_n - 1\right)$$

$$H^{Sus}_n(X) := H_n \,\mathrm{Cor}(\Delta^{\bullet}_{alg}, X)$$

More generally

$$H^{Sus}_{n,m}(X) := H_{n-m}\, \mathrm{Cor}\left(\Delta^{\bullet}_{alg} \times \mathbb{G}_m^{\wedge m}, X\right)$$

$m \geq 0$

Obviously: this is an algebraic version of singular homology.

$$k \subset \mathbb{C}, \qquad H^{Sus}_{n,m}(X) \to H^{Sing}_n(X)(-m)$$

<u>Remark</u>: If X is smooth and proper then Suslin homology is just motivic

Cohomology $\qquad d = \dim X$

$$H^{\mathrm{Sing}}_{n,m}(X) = H^{2d-n,\, d-m}_{\mathrm{mot}}(X).$$

Remark:

$$\begin{array}{ccc} H^{\mathrm{Sus}}_{n,m}(X) & \longrightarrow & H^{\mathrm{Sing}}_{n}(X)(-m) \\ \Downarrow & & \wr\!\wr \\ \alpha & \longmapsto & \gamma \in H^{\mathrm{Sing}}_{n}(X) \end{array} \qquad \underline{k = \mathbb{Q}}$$

$\omega \in H^n_{dR}(X)$, $\int_\gamma \omega \overset{?}{=} \mathbb{Q} \cdot (2\pi i)^m.$

Construction: If K a complete non-arch. fields (eg $\mathbb{Q}_p$) and X a rigid analytic K-variety, then

$$\mathrm{Cor}(\Delta_{rig}, X)$$

$$\Delta^n_{rig} = \mathrm{Spa}\left(K\{t_0, \dots, t_n\} / t_0 + \dots + t_n - 1 \right)$$

$$H^{Sus}_{n,m}(X) = H_{n-m}(\Delta'_{rig} \times \partial B^{\wedge m}, X)$$

By construction, $X / \mathbb{Q}$

$$H^{sus}_{n,m}(X) \longrightarrow H^{Sus}_{n,m}(X^{an})$$

$$\mathbb{Q}_p$$

Proposition: $X / \mathbb{Q}$,

$\omega \in H^n_{dR}(X)(m)$, $\gamma \in H^{Sus}_{n,m}(X^{an})$

then $\exists \int_\gamma \omega \in \mathbb{Q}_p$.

Rmk $\alpha \in H^{Sus}_{n,m}(X) \longmapsto \gamma \in H^{Sus}_{n,m}(X^{an})$

$\int_\gamma \omega \in \mathbb{Q}$.

Example: X is smooth and projective with good reduction at p

$$H_{n,m}(X^{an}) = H_{n,m}(\mathcal{X}_p)$$

III Motivic framework for p-adic periods

over $\mathbb{C}$: we have the notion of "abstract period" (à Kontsevich – Zagier).

these form an algebra $\mathcal{P}$, which is the algebra of a torsor over the motivic Galois group. Also $\mathcal{P} \longrightarrow \mathbb{C}$.

conjectured to be

injective.

One way to introduce $\mathcal{P}$ is as follows:

DM = Voevodsky cat of motives over $\mathbb{Q}$.

$$B : DM \xrightarrow{\text{Betti realisation}} D(\mathbb{Q})$$

$\cup$

Γ_{dR} — the object representing de Rham cohomology.

$\Omega^{\cdot}_{/k}$ viewed as a complex of presheaves on $Sm_{\mathbb{Q}}$.

$$\mathcal{P} = B(\Pi_{dR})$$

__Fact:__ $\mathcal{P}$ is connective i.e, $H_i(\mathcal{P}) = 0$ $(i < 0)$

1) $G_{mot} \curvearrowright Spec(\mathcal{P})$ and this is torsor.

2) $\mathcal{P} \to \mathbb{C}$.

$\exists \quad \Pi_B$ rep. Betti coh.

$$\Pi_{dR} \otimes \mathbb{C} \simeq \Pi_B \otimes \mathbb{C}.$$

$$\Rightarrow B(\Pi_{dR}) \to B(\Pi_{dR} \otimes \mathbb{C})$$

$$\simeq B(\Pi_B \otimes \mathbb{C})$$

$$\simeq \underbrace{B(\Pi_B)}_{= \mathcal{O}(G_{mot})} \otimes \mathbb{C} \xrightarrow{ev_1} \mathbb{C}$$

Over $\mathbb{Q}_p$:

$$\mathrm{Rig}: \quad DM(\mathbb{Q}) \longrightarrow \mathrm{RigDM}(\mathbb{Q}_p)$$

$$M(X) \longmapsto M(X^{an})$$

$$\mathrm{Rig}(\Pi_{dR}) \in \mathrm{RigDM}(\mathbb{Q}_p).$$

thm: $\mathrm{Rig}(\Pi_{dR})$ represents a Weil cohomology theory on rigid analytic varieties. (It has a Kunneth formula).

Rmk We get in this way a new

Weil coh. theory.

1) Coefficient ring is very big.

the ring of abstract p-adic periods $\mathcal{P}_p$

2) this Weil coh. theory compares with all

the classical ones:

$\ell \neq p$, $\exists$ $\mathcal{P}_p \longrightarrow \overline{\mathbb{Q}_\ell}$ and

an iso $\left(\mathrm{Rig}\,\Pi_{dR}\right) \underset{\mathcal{P}_p}{\otimes} \overline{\mathbb{Q}_\ell} \simeq \Pi_\ell \underset{\mathbb{Q}_\ell}{\otimes} \overline{\mathbb{Q}_\ell}$

Conclusion: $\exists$ dg-algebra $\mathcal{P}_p$, connective, and $\mathcal{P}_p \longrightarrow \mathbb{A}_p$,

and an action of $G_{mot} \curvearrowright \mathcal{P}_p$.

⚠ $\mathrm{Spec}(\mathcal{P}_p)$ is not a torsor over

G_{mot}.

IV) Connectivity.

Q) Is the motivic Hopf alg connective?

ch 0 ✓.

ch p ?

Defn dg-Hopf algebra =

cosimplicial dg-algebra $\mathcal{H}^\bullet$

1) $\mathcal{H}^0$ = base ring / field.

2) $\mathcal{H}^1 \otimes \cdots \otimes \mathcal{H}^1 \xrightarrow{q.\text{iso}} \mathcal{H}^n$ $\quad \{i, i+1\} \subset \Delta^n$

Defn: let Γ_W be a Weil coh. theory

$(\Gamma_W : Sm_{\mathbb{Q}} \longrightarrow dg\text{-alg})$.

then $\Pi_W \in DM$.

$$\Pi_W \otimes \Pi_W \simeq \underline{\Pi_W \otimes \overbrace{(dg\_algebra)}^{\mathcal{H}_{mot}(\Gamma_W)}}$$

$$\oplus \, \Pi_W[i]$$

thm: $\Gamma_W = \Gamma_\ell$ (ℓ-adic coh)

then $\mathcal{H}_{mot}(\Gamma_\ell)$ is connective.

(**Proof**: one reduces to show that

$$\mathcal{H}_{mot}\left(Rig(T_{dR})\right) \text{ is connective}$$

$$Rig(T_{dR}) \overset{comp}{\dashrightarrow} T_\ell \otimes \overline{\mathbb{Q}}_\ell .$$

$\rightarrow$ $G_{mot}(T_\ell)$ is the stabilizer of $G_{mot}(T_{dR})$ acting on

$$\begin{array}{c} Spec(\mathcal{P}_p) \\ \downarrow \\ Spec(\overline{\mathbb{Q}}_\ell) \end{array}$$

$\boxtimes$)

References

- Connectivity of the motivic Hopf algebras.
- Nouvelles cohomologies de Weil
- Ancona-Fratila

- André