About computer regularizations of divergent polyzetas[1]

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INTRODUCTION

Starting with Abel's theorem and E-M summation formula

For $s \in \mathbb{C}, |z| < 1$, the following functions are well defined[2]

$$\mathrm{Li}_s(z) := \sum_{n>0} \frac{z^n}{n^s} \quad \text{and} \quad \frac{1}{(1-z)} \mathrm{Li}_s(z) = \sum_{n\geq 0} \mathrm{H}_s(n) z^n.$$

In particular, $\mathrm{Li}_1(z) = -\log(1-z)$ and then

$$\frac{1}{1-z}\mathrm{Li}_1(z) = \frac{1}{1-z} \log \frac{1}{1-z} = \sum_{n\geq 0} \mathrm{H}_1(n) z^n.$$

For $\Re(s) > 1$, applying a theorem by Abel, one has

$$\lim_{z\to 1} \mathrm{Li}_s(z) = \lim_{n\to+\infty} \mathrm{H}_s(n) = \sum_{n>0} \frac{1}{n^s} < +\infty.$$

This common limit is denoted usually by $\zeta(s)$, the Riemann ζ function.

For any $r \in \mathbb{N}_+$, by the Euler-Maclaurin summation formula, one has[3]

$$\begin{aligned} r \geq 2, \quad \mathrm{H}_r(n) &= \zeta(r) - \sum_{j=r-1}^{k-1} \frac{B_{j-r+1}}{j} \binom{j}{r-1} \frac{1}{n^j} + O\Big(\frac{1}{n^k}\Big), \\ r = 1, \quad \mathrm{H}_1(n) &= \log n + \gamma - \sum_{j=1}^{k-1} \frac{B_j}{j} \frac{1}{n^j} + O\Big(\frac{1}{n^k}\Big). \end{aligned}$$

This means that, using the comparison scale $\{n^a \log^b(n)\}_{a\in\mathbb{Z}_{\leq -1}, b\in\mathbb{N}}$,

$$\gamma_k := \underset{n\to+\infty}{\mathrm{f.p.}} \mathrm{H}_k(n) = \begin{cases} \gamma & \text{if} \quad k = 1, \\ \zeta(k) & \text{if} \quad k > 1. \end{cases}$$

[2] $\mathrm{H}_s : \mathbb{N} \longrightarrow \mathbb{Q}$, $n \longmapsto \mathrm{H}_s(n) = 1 + 2^{-s} + \ldots + n^{-s}$.

[3] B_j is the jth Bernoulli number and γ is the Euler-Mascheroni constant.

From ζ function in several variables to monoids

For $(s_1,\ldots,s_r)\in\mathbb{C}^r, |z|< 1$, the following functions are well defined[4]

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) := \sum_{n_1>\ldots>n_r>0} \frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}} \quad\text{and}\quad \frac{\mathrm{Li}_{s_1,\ldots,s_r}(z)}{(1-z)} = \sum_{n\geq 0} \mathrm{H}_{s_1,\ldots,s_r}(n) z^n.$$

Over $\mathcal{H}_r = \{(s_1,\ldots,s_r)\in\mathbb{C}^r | \forall m=1,\ldots,r, \Re(s_1)+\ldots+\Re(s_m) > m\}$,

$$\lim_{z\to 1} \mathrm{Li}_{s_1,\ldots,s_r}(z) = \lim_{n\to+\infty} \mathrm{H}_{s_1,\ldots,s_r}(n) = \sum_{n_1>\ldots>n_r>0} n_1^{-s_1}\ldots n_r^{-s_r} < +\infty.$$

This common limit can be denoted by $\zeta_r(s_1,\ldots,s_r)$ or by $\zeta(s_1,\ldots,s_r)$.

$$\mathcal{Z} := \mathrm{span}_{\mathbb{Q}}\{\mathrm{Li}_{s_1,\ldots,s_r}(1)\}_{\substack{s_1,\ldots,s_r\in\mathbb{N}_+\\ s_1>1, r>0}} = \mathrm{span}_{\mathbb{Q}}\{\mathrm{H}_{s_1,\ldots,s_r}(+\infty)\}_{\substack{s_1,\ldots,s_r\in\mathbb{N}_+\\ s_1>1, r>0}},$$

$$(s_1,\ldots,s_r)\in\mathbb{N}_+^r \leftrightarrow y_{s_1}\ldots y_{s_r} \in Y^* \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1 \in X^*x_1,$$

where $X=\{x_0,x_1\}$ (resp. $Y=\{y_k\}_{k\geq 1}$) generating the free monoid $(X^*, 1_{X^*})$ (resp. $(Y^*, 1_{Y^*})$).

Let $\mathcal{L}ynX$ (resp. $\mathcal{L}ynY$) be the ordered set of Lyndon words over X (resp. Y) with $x_1\succ x_0$ (resp. $y_1\succ y_2\succ\cdots$). Then (Perrin's lemma)

$$l\in\mathcal{L}ynY \iff \pi_X l\in\mathcal{L}ynX\setminus\{x_0\}.$$

In all the sequel, let $\mathcal{X}$ denote X or Y.

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[4] $\mathrm{H}_{s_1,\ldots,s_r}:\mathbb{N}\longrightarrow\mathbb{Q}$, $N\longmapsto \mathrm{H}_{s_1,\ldots,s_r}(N) = \sum_{n_1>\ldots>n_r>0}^{N} n_1^{-s_1}\ldots n_r^{-s_r}$.

ALBEBRAIC COMBINATORICS ON NONCOMMUTATIVE POWER SERIES

MRS[10] factorization of ⧢-group-like series in $\mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$

$(\mathbb{C}\langle\mathcal{X}\rangle, \mathtt{conc})$[5] and $(\mathbb{C}\langle\langle\mathcal{X}\rangle\rangle, \mathtt{conc})$ (resp. $\mathcal{L}ie_{\mathbb{C}}\langle\mathcal{X}\rangle$ and $\mathcal{L}ie_{\mathbb{C}}\langle\langle\mathcal{X}\rangle\rangle$): algebras (resp. Lie algebras) of polynomials and of series over $\mathcal{X}$ with coeff. in $\mathbb{C}$.

Let[6] S be a ⧢-group-like series[7] in $\mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$. Then[8]

$$S = \sum_{w\in\mathcal{X}^*} \langle S|w\rangle w = \sum_{w\in\mathcal{X}^*} \langle S|S_w\rangle P_w = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{\langle S|S_l\rangle P_l},$$

where $\begin{matrix}\{P_l\}_{l\in\mathcal{L}yn\mathcal{X}}\\ \{S_l\}_{l\in\mathcal{L}yn\mathcal{X}}\end{matrix}$ is basis of[9] $\begin{matrix}\mathcal{L}ie_{\mathbb{Q}}\langle\mathcal{X}\rangle\\ (\mathbb{Q}\langle\mathcal{X}\rangle, ⧢)\end{matrix}$, over which is constructed the pair of dual bases $\begin{matrix}\{P_w\}_{w\in\mathcal{X}^*}\\ \{S_w\}_{w\in\mathcal{X}^*}\end{matrix}$ which are defined, for $w = l_1^{i_1}\ldots l_k^{i_k}$ ($l_1,\ldots,l_k \in \mathcal{L}yn\mathcal{X}$ and $l_1 > \ldots > l_k$), as follows

$$P_w = P_{l_1}^{i_1}\ldots P_{l_k}^{i_k} \text{ and } S_w = \frac{1}{i_1!\ldots i_k!} S_{l_1}^{⧢\, i_1} ⧢ \ldots ⧢ S_{l_k}^{⧢\, i_k}.$$

$$\langle P_l|S_k\rangle = \delta_{l,k} \text{ (for } l,k \in \mathcal{L}yn\mathcal{X}) \text{ and } \langle P_u|S_v\rangle = \delta_{u,v} \text{ (for } u,v \in \mathcal{X}^*).$$

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[5] $\Delta_{\mathtt{conc}}$ is defined by $\Delta_{\mathtt{conc}}(w) = \sum_{u,v\in\mathcal{X}^*, uv=w} u\otimes v$.

[6] ⧢ is defined, for any $x,y\in\mathcal{X}$ and $w,v\in\mathcal{X}^*$, by $w ⧢ 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} ⧢ w = w$ and $xw ⧢ yv = x(w ⧢ yv) + y(xw ⧢ v)$. Or equivalently, $\Delta_{⧢}(x) = 1_{\mathcal{X}^*}\otimes x + x\otimes 1_{\mathcal{X}^*}$.

[7] *i.e.* $\Delta_{⧢}(S) = S\otimes S$ and $\langle S|1_{\mathcal{X}^*}\rangle = 1$

[8] $\{\langle S|S_l\rangle\}_{l\in\mathcal{L}yn\mathcal{X}}$ are called locale coordinates (of second kind) of S on the group of ⧢-group-like series.

[9] $P\in\mathcal{L}ie_{\mathbb{Q}}\langle\mathcal{X}\rangle \iff \Delta_{⧢}(P) = 1_{\mathcal{X}^*}\otimes P + P\otimes 1_{\mathcal{X}^*}$.

[10] MRS is an contraction of Mélançon, Reutenauer and Schützenberger.

Computing examples of $\{P_l\}_{l\in\mathcal{L}yn\mathcal{X}}$, $\{S_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ in $(\mathbb{Q}\langle\mathcal{X}\rangle, \mathtt{conc}, 1_{\mathcal{X}^*}, \Delta_{\sqcup\!\sqcup})$

The polynomials $\{P_l\}_{l\in\mathcal{L}yn\mathcal{X}}$, $\{S_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ are homogenous in weight[11].

l	P_l	S_l
x_0	x_0	x_0
x_1	x_1	x_1
x_0x_1	$[x_0, x_1]$	x_0x_1
$x_0^2x_1$	$[x_0, [x_0, x_1]]$	$x_0^2x_1$
$x_0x_1^2$	$[[x_0, x_1], x_1]$	$x_0x_1^2$
$x_0^3x_1$	$[x_0, [x_0, [x_0, x_1]]]$	$x_0^3x_1$
$x_0^2x_1^2$	$[x_0, [[x_0, x_1], x_1]]$	$x_0^2x_1^2$
$x_0x_1^3$	$[[[x_0, x_1], x_1], x_1]$	$x_0x_1^3$
$x_0^4x_1$	$[x_0, [x_0, [x_0, [x_0, x_1]]]]$	$x_0^4x_1$
$x_0^3x_1^2$	$[x_0, [x_0, [[x_0, x_1], x_1]]]$	$x_0^3x_1^2$
$x_0^2x_1x_0x_1$	$[[x_0, [x_0, x_1]], [x_0, x_1]]$	$2x_0^3x_1^2 + x_0^2x_1x_0x_1$
$x_0^2x_1^3$	$[x_0, [[[x_0, x_1], x_1], x_1]]$	$x_0^2x_1^3$
$x_0x_1x_0x_1^2$	$[[x_0, x_1], [[x_0, x_1], x_1]]$	$3x_0^2x_1^3 + x_0x_1x_0x_1^2$
$x_0x_1^4$	$[[[[x_0, x_1], x_1], x_1], x_1]$	$x_0x_1^4$
$x_0^5x_1$	$[x_0, [x_0, [x_0, [x_0, [x_0, x_1]]]]]$	$x_0^5x_1$
$x_0^4x_1^2$	$[x_0, [x_0, [x_0, [[x_0, x_1], x_1]]]]$	$x_0^4x_1^2$
$x_0^3x_1x_0x_1$	$[x_0, [[x_0, [x_0, x_1]], [x_0, x_1]]]$	$2x_0^4x_1^2 + x_0^3x_1x_0x_1$
$x_0^3x_1^3$	$[x_0, [x_0, [[[x_0, x_1], x_1], x_1]]]$	$x_0^3x_1^3$
$x_0^2x_1x_0x_1^2$	$[x_0, [[x_0, x_1], [[x_0, x_1], x_1]]]$	$3x_0^3x_1^3 + x_0^2x_1x_0x_1^2$
$x_0^2x_1^2x_0x_1$	$[[x_0, [[x_0, x_1], x_1]], [x_0, x_1]]$	$6x_0^3x_1^3 + 3x_0^2x_1x_0x_1^2 + x_0^2x_1^2x_0x_1$
$x_0^2x_1^4$	$[x_0, [[[[x_0, x_1], x_1], x_1], x_1]]$	$x_0^2x_1^4$
$x_0x_1x_0x_1^3$	$[[x_0, x_1], [[[x_0, x_1], x_1], x_1]]$	$4x_0^2x_1^4 + x_0x_1x_0x_1^3$
$x_0x_1^5$	$[[[[[x_0, x_1], x_1], x_1], x_1], x_1]$	$x_0x_1^5$

[11] The weight of $w \in \mathcal{X}^*$ is the length of w, *i.e.* $|w|$.
For any $l \in \mathcal{L}yn\mathcal{X}$, the weight of P_l equals the weight of S_l and equals $|l|$.

MRS factorization of $⧢$-group-like series in $\mathbb{C}\langle\langle Y\rangle\rangle$

Let[12] S be a $⧢$-group-like series[13] in $\mathbb{C}\langle\langle Y\rangle\rangle$. Then[14]

$$S=\sum_{w\in Y^*}\langle S|w\rangle w=\sum_{w\in Y^*}\langle S|\Sigma_w\rangle\Pi_w=\prod_{l\in\mathcal{L}ynY}^{\searrow}e^{\langle S|\Sigma_l\rangle\Pi_l},$$

where $\frac{\{\Pi_l\}_{l\in\mathcal{L}ynY}}{\{\Sigma_l\}_{l\in\mathcal{L}ynY}}$ is basis of[15] $\frac{\mathrm{Prim}_{⧢}(Y)}{(\mathbb{Q}\langle Y\rangle,⧢)}$, over which is constructed the pair of dual bases $\frac{\{\Pi_w\}_{w\in Y^*}}{\{\Sigma_w\}_{w\in Y^*}}$ which are defined, for $w=l_1^{i_1}\ldots l_k^{i_k}$ $(l_1,\ldots,l_k\in\mathcal{L}ynY$ and $l_1>\ldots>l_k)$, as follows

$$\Pi_w=\Pi_{l_1}^{i_1}\ldots\Pi_{l_k}^{i_k}\text{ and }\Sigma_w=\frac{1}{i_1!\ldots i_k!}\Sigma_{l_1}^{⧢ i_1}⧢\ldots⧢\Sigma_{l_k}^{⧢ i_k}.$$

$$\langle\Pi_l|\Sigma_k\rangle=\delta_{l,k}\ (\text{for } l,k\in\mathcal{L}ynY)\text{ and }\langle\Pi_u|\Sigma_v\rangle=\delta_{u,v}\ (\text{for } u,v\in Y^*).$$

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[12] $⧢$ is defined, for any $y_i,y_j\in Y$ and $w,v\in Y^*$, by $w⧢1_{Y^*}=1_{Y^*}⧢w=w$ and $y_iu⧢y_jv=y_i(u⧢y_jv)+y_j(y_iu⧢v)+y_{i+j}(u⧢v)$. Or equivalently, $\Delta_{⧢}(y_i)=1_{Y^*}\otimes y_i+y_i\otimes 1_{Y^*}$.

[13] *i.e.* $\Delta_{⧢}(S)=S\otimes S$ and $\langle S|1_{Y^*}\rangle=1$.

[14] $\{\langle S|\Sigma_l\rangle\}_{l\in\mathcal{L}ynY}$ are called locale coordinates (of second kind) of S on the group of $⧢$-group-like series.

[15] $P\in\mathrm{Prim}_{⧢}(Y)\iff\Delta_{⧢}(P)=1_{Y^*}\otimes P+P\otimes 1_{Y^*}$.

Computing examples of $\{\Pi_l\}_{l\in\mathcal{L}ynY}$ $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ in $(\mathbb{Q}\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\uplus})$

The polynomials $\{\Pi_l\}_{l\in\mathcal{L}ynY}$ $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ are homogenous in weight[16].

l	Π_l	Σ_l
y_2	$y_2 - \frac{1}{2}y_1^2$	y_2
y_3	$y_3 - \frac{1}{2}y_1y_2 - \frac{1}{2}y_2y_1 + \frac{1}{3}y_1^3$	y_3
y_2y_1	$y_2y_1 - y_1y_2$	$\frac{1}{2}y_3 + y_2y_1$
y_4	$y_4 - \frac{1}{2}y_1y_3 - \frac{1}{2}y_2^2 - \frac{1}{2}y_3y_1 + \frac{1}{3}y_1^2y_2 + \frac{1}{3}y_1y_2y_1 + \frac{1}{3}y_2y_1^2 - \frac{1}{4}y_1^4$	y_4
y_3y_1	$y_3y_1 - \frac{1}{2}y_2y_1^2 - y_1y_3 + \frac{1}{2}y_1^2y_2$	$\frac{1}{2}y_4 + y_3y_1$
$y_2y_1^2$	$y_2y_1^2 - 2\,y_1y_2y_1 + y_1^2y_2$	$\frac{1}{6}y_4 + \frac{1}{2}y_3y_1 + \frac{1}{2}y_2^2 + y_2y_1^2$
y_5	$y_5 - \frac{1}{2}y_1y_4 - \frac{1}{2}y_2y_3 - \frac{1}{2}y_3y_2 - \frac{1}{2}y_4y_1 + \frac{1}{3}y_1^2y_3 + \frac{1}{3}y_1y_2^2 + \frac{1}{3}y_1y_3y_1 + \frac{1}{3}y_2y_1y_2 + \frac{1}{3}y_2^2y_1 + \frac{1}{3}y_3y_1^2 - \frac{1}{4}y_1^3y_2 - \frac{1}{4}y_1^2y_2y_1 - \frac{1}{4}y_1y_2y_1^2 - \frac{1}{4}y_2y_1^2y_1 + \frac{1}{5}y_1^4y_1$	y_5
y_4y_1	$y_4y_1 - \frac{1}{2}y_2^2y_1 - \frac{1}{2}y_3y_1^2 + \frac{1}{3}y_2y_1^3 - y_1y_4 + \frac{1}{2}y_1^2y_3 + \frac{1}{2}y_1y_2^2 - \frac{1}{3}y_1^3y_2$	$\frac{1}{2}y_5 + y_4y_1$
y_3y_2	$y_3y_2 - \frac{1}{2}y_3y_1^2 - \frac{1}{2}y_1y_2^2 + \frac{1}{4}y_1y_2y_1^2 - \frac{1}{12}y_2y_1^3 + \frac{1}{12}y_1^3y_2 - y_2y_3 + \frac{1}{2}y_2^2y_1 + \frac{1}{2}y_1^2y_3 - \frac{1}{4}y_1^2y_2y_1$	$\frac{1}{2}y_5 + y_3y_2$
$y_3y_1^2$	$y_3y_1^2 - \frac{1}{2}y_2y_1^3 - 2y_1y_3y_1 + \frac{1}{2}y_1^2y_2y_1 + \frac{1}{2}y_1y_2y_1^2 + y_1^2y_3 - \frac{1}{2}y_1^3y_2$	$\frac{1}{6}y_5 + \frac{1}{2}y_4y_1 + \frac{1}{2}y_3y_2 + y_3y_1^2$
$y_2^2y_1$	$y_2^2y_1 - 2y_2y_1y_2 - \frac{1}{2}y_1^2y_2y_1 + \frac{1}{2}y_1^3y_2 + \frac{1}{2}y_2y_1^3 + y_1y_2^2 - \frac{1}{2}y_1y_2y_1^2$	$\frac{1}{6}y_5 + \frac{1}{2}y_4y_1 + \frac{1}{2}y_2y_3 + y_2^2y_1$
$y_2y_1^3$	$y_2y_1^3 - 3y_1y_2y_1^2 + 3y_1^2y_2y_1 - y_1^3y_2$	$\frac{1}{24}y_5 + \frac{1}{6}y_4y_1 + \frac{1}{4}y_3y_2 + \frac{1}{2}y_3y_1^2 + \frac{1}{6}y_2y_3 + \frac{1}{2}y_2^2y_1 + \frac{1}{2}y_2y_1y_2 + y_2y_1^3$

[16]The weight of $w = y_{s_1}\ldots y_{s_r} \in Y^*$ is the number $(w) = s_1 + \ldots + s_r$.
For any $l \in \mathcal{L}ynY$, the weight of Π_l equals the weight of Σ_l and equals (l).

Representative series and Sweedler's dual

$\mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ is the smallest algebra containing $\mathbb{C}\langle\mathcal{X}\rangle$, closed by[17] $\{+, \mathrm{conc}, *\}$. It is also closed by $\sqcup\!\sqcup$ and, in addition, $\mathbb{C}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$ is also closed by $⧢$ (stuffle).

Proposition

The following assertions are equivalent

1. $S \in \mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$.
2. *There is a lin. rep., (ν, μ, η) of rank n, of S with $\nu \in M_{1,n}(\mathbb{C})$, $\eta \in M_{n,1}(\mathbb{C})$ and a morphism of monoids $\mu : \mathcal{X}^* \to M_{n,n}(\mathbb{C})$ s.t.*
$$S = \nu\Big(\prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{S_l \mu(P_l)}\Big)\eta \quad \Big(\text{and also} \quad S = \nu\Big(\prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\Sigma_l \mu(\Pi_l)}\Big)\eta\Big).$$
3. *The shifts[18] $\{S \triangleleft w\}_{w\in\mathcal{X}^*}$ (resp. $\{w \triangleright S\}_{w\in\mathcal{X}^*}$) lie within a finitely generated shift-invariant $\mathbb{C}$-module.*
4. $\exists\{G_i, D_i\}_{i\in I \text{ finite}}$ *in* $\begin{matrix}\mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle\times\mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \\ \mathbb{C}^{\mathrm{rat}}\langle\langle Y\rangle\rangle\times\mathbb{C}^{\mathrm{rat}}\langle\langle Y\rangle\rangle\end{matrix}$ *s.t.* $\Delta_{\mathrm{conc}}(S) = \sum_{i\in I} G_i \otimes D_i$.

Hence, $\begin{matrix}\mathcal{H}^{\circ}_{\sqcup\!\sqcup}(\mathcal{X}) \cong (\mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*}, \Delta_{\mathrm{conc}}), \\ \mathcal{H}^{\circ}_{⧢}(Y) \cong (\mathbb{C}^{\mathrm{rat}}\langle\langle Y\rangle\rangle, ⧢, 1_{Y^*}, \Delta_{\mathrm{conc}}).\end{matrix}$

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[17] The Kleene star of S, $\langle S|1_{\mathcal{X}^*}\rangle = 0$, is the sum $S^* = \sum_{n\geq 0} S^n$.

[18] *Left* (resp. *right*) shift of S by $P \in \mathbb{C}\langle\mathcal{X}\rangle$ is defined, for any $w \in \mathcal{X}^*$, by
$$\langle P \triangleright S|w\rangle = \langle S|wP\rangle \quad (\text{resp.} \quad \langle S \triangleleft P|w\rangle = \langle S|Pw\rangle).$$

Linear representations of rational series

Proposition

The module $\mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ *(resp.* $\mathbb{C}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$*) is closed by* $⧢$ *(resp.* $⊎$*). Moreover, for any* $i=1,2$*, let* $R_i\in\mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ *and* (ν_i,μ_i,η_i) *be its representation of dimension* n_i*. Then the linear representation of*

$$\begin{array}{rcl}
R_i^* & \text{is} & \left(\begin{pmatrix} 0 & 1 \end{pmatrix}, \left\{ \begin{pmatrix} \mu_i(x)+\eta_i\nu_i\mu_i(x) & 0 \\ \nu_i\eta_i & 0 \end{pmatrix} \right\}_{x\in\mathcal{X}}, \begin{pmatrix} \eta_i \\ 1 \end{pmatrix} \right), \\
\text{that of } R_1+R_2 & \text{is} & \left(\begin{pmatrix} \nu_1 & \nu_2 \end{pmatrix}, \left\{ \begin{pmatrix} \mu_1(x) & \mathbf{0} \\ \mathbf{0} & \mu_2(x) \end{pmatrix} \right\}_{x\in\mathcal{X}}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right), \\
\text{that of } R_1R_2 & \text{is} & \left(\begin{pmatrix} \nu_1 & 0 \end{pmatrix}, \left\{ \begin{pmatrix} \mu_1(x) & \eta_1\nu_2\mu_2(x) \\ 0 & \mu_2(x) \end{pmatrix} \right\}_{x\in\mathcal{X}}, \begin{pmatrix} \eta_1\mu_2\eta_2 \\ \eta_2 \end{pmatrix} \right), \\
\text{that of } R_1 ⧢ R_2 & \text{is} & (\nu_1\otimes\nu_2, \{\mu_1(x)\otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1}\otimes\mu_2(x)\}_{x\in\mathcal{X}}, \eta_1\otimes\eta_2), \\
\text{that of } R_1 ⊎ R_2 & \text{is} & (\nu_1\otimes\nu_2, \{\mu_1(y_k)\otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1}\otimes\mu_2(y_k) \\
 & & + \sum_{i+j=k} \mu_1(y_i)\otimes\mu_2(y_j)\}_{k\geq 1}, \eta_1\otimes\eta_2).
\end{array}$$

Kleene stars of the plane

$$\Big(\sum_{x\in\mathcal{X}}\alpha_x x\Big)^* \sqcup\!\sqcup \Big(\sum_{x\in\mathcal{X}}\beta_x x\Big)^* = \Big(\sum_{x\in\mathcal{X}}(\alpha_x+\beta_x)x\Big)^*,$$
$$\Big(\sum_{s\geq 1}a_s y_s\Big)^* \uplus \Big(\sum_{s\geq 1}b_s y_s\Big)^* = \Big(\sum_{s\geq 1}(a_s+b_s)y_s + \sum_{r,s\geq 1}a_s b_r y_{s+r}\Big)^*,$$

where $\{\alpha_x\}_{x\in\mathcal{X}},\{\beta_x\}_{x\in\mathcal{X}},\{a_s\}_{s\geq 1},\{b_s\}_{s\geq 1}$ are complex numbers. Then

1. Let $y_s, y_r \in Y, a_s, a_r \in \mathbb{C}$. Then
$$(-a_s y_s)^* \uplus (a_s y_s)^* = (-a_s^2 y_{2s})^*,$$
$$(a_s y_s)^* \uplus (a_r y_r)^* = (a_s y_s + a_r y_r + a_s a_r y_{s+r})^*.$$

2. $\forall n\in\mathbb{N}, c\in\mathbb{C}, x\in\mathcal{X}$, $\langle (cx)^* \sqcup\!\sqcup (1+cx)^n | (cx)^k\rangle = \binom{n+k}{k}, k\in\mathbb{N}$.

3. Let $x\in\mathcal{X}, y_k\in Y$ and $c\in\mathbb{C}, n\in\mathbb{N}_{\geq 1}$. Then
$$((cx)^*)^{\sqcup\!\sqcup\, n} = (ncx)^*, \quad ((cx)^*)^n = (cx)^* \sqcup\!\sqcup (1+cx)^{n-1},$$
$$((cy_k)^*)^{\uplus\, n} = \Big(\sum_{i=1}^{n}(n-i+1)cy_{ik}\Big)^* = \mathop{\sqcup\!\sqcup}_{i=1}^{n}((n-i+1)cy_{ik})^*.$$

4. Let $m, n_1,\ldots,n_k\in\mathbb{N}, c_1,\ldots,c_k\in\mathbb{C}$ and $x_1,\ldots,x_m\in\mathcal{X}$.Then
$$\mathop{\sqcup\!\sqcup}_{i=1}^{m}((c_i x_i)^*)^{n_i+1} = \mathop{\sqcup\!\sqcup}_{i=1}^{m}(c_i x_i)^* \sqcup\!\sqcup (1+c_i x_i)^{n_i} = \Big(\sum_{i=1}^{m}c_i x_i\Big)^* \sqcup\!\sqcup \mathop{\sqcup\!\sqcup}_{i=1}^{m}(1+c_i x_i)^{n_i},$$
$$\forall k\in\mathbb{N}, \quad \langle \mathop{\sqcup\!\sqcup}_{i=1}^{m}((c_i x_i)^*)^{n_i+1} | \Big(\sum_{i=1}^{m}c_i x_i\Big)^{\sqcup\!\sqcup\, k}\rangle = \binom{n_1+\cdots+n_m}{k}.$$

Linear representation of $(-t^2x_0x_1)^* \sqcup\!\sqcup (t^2x_0x_1)^*$

$$
\begin{aligned}
(t^2x_0x_1)^* &\leftrightarrow (\nu_1,\{\mu_1(x_0),\mu_1(x_1)\},\eta_1),\\
(-t^2x_0x_1)^* &\leftrightarrow (\nu_2,\{\mu_2(x_0),\mu_2(x_1)\},\eta_2),
\end{aligned}
$$

$$
\nu_1=\begin{pmatrix}1 & 0\end{pmatrix},\quad \eta_1=\begin{pmatrix}1\\0\end{pmatrix},\quad \mu_1(x_0)=\begin{pmatrix}0 & t\\0 & 0\end{pmatrix},\quad \mu_1(x_1)=\begin{pmatrix}0 & 0\\t & 0\end{pmatrix},
$$

$$
\nu_2=\begin{pmatrix}1 & 0\end{pmatrix},\quad \eta_2=\begin{pmatrix}1\\0\end{pmatrix},\quad \mu_2(x_0)=\begin{pmatrix}0 & \mathrm{i}t\\0 & 0\end{pmatrix},\quad \mu_2(x_1)=\begin{pmatrix}0 & 0\\\mathrm{i}t & 0\end{pmatrix}.
$$

$$
\begin{aligned}
(\nu,\{\mu(x_0),\mu(x_1)\},\eta) = (\nu_1\otimes\nu_2,\{&\mu_1(x_0)\otimes \mathrm{I}_2+\mathrm{I}_2\otimes\mu_2(x_0),\\
&\mu_1(x_1)\otimes \mathrm{I}_2+\mathrm{I}_2\otimes\mu_2(x_1)\},\eta_1\otimes\eta_2),
\end{aligned}
$$

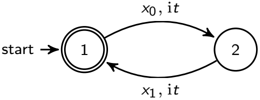
$$
\nu=\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix},
$$

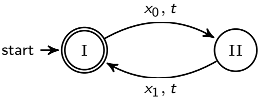
$$
\mu(x_0)=\begin{pmatrix}0&0&t&0\\0&0&0&t\\0&0&0&0\\0&0&0&0\end{pmatrix}+\begin{pmatrix}0&\mathrm{i}t&0&0\\0&0&0&0\\0&0&0&\mathrm{i}t\\0&0&0&0\end{pmatrix}=\begin{pmatrix}0&\mathrm{i}t&t&0\\0&0&0&t\\0&0&0&\mathrm{i}t\\0&0&0&0\end{pmatrix},
$$

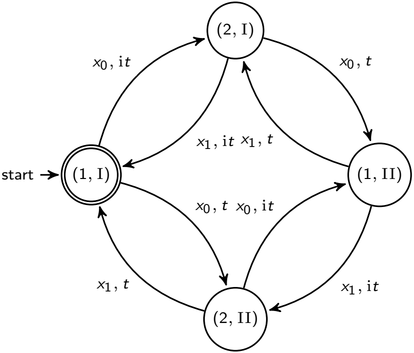
$$
\mu(x_1)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\t&0&0&0\\0&t&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&0\\\mathrm{i}t&0&0&0\\0&0&0&0\\0&0&\mathrm{i}t&0\end{pmatrix}=\begin{pmatrix}0&0&0&0\\\mathrm{i}t&0&0&0\\t&0&0&0\\0&t&\mathrm{i}t&0\end{pmatrix},\eta=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}.
$$

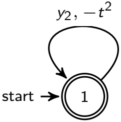
Identities

$$(-t^2x_0x_1)^* \sqcup\!\sqcup (t^2x_0x_1)^* = (-4t^4x_0^2x_1^2)^*$$
$$(-t^2y_2)^* \uplus (t^2y_2)^* = (-t^4y_4)^*$$

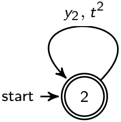




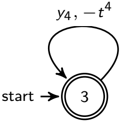




$(-t^2y_2)^* \leftrightarrow (\nu_2, \mu_2(y_2), \eta_2)$
$= (1, -t^2, 1),$



$(t^2y_2)^* \leftrightarrow (\nu_1, \mu_1(y_2), \eta_1)$
$= (1, t^2, 1),$



$(-t^4y_4)^* \leftrightarrow (\nu, \mu(y_4), \eta)$
$= (1, -t^4, 1).$

Subalgebra of $\mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$

Proposition

1. *The algebras* $(\mathbb{C}[\{x^*\}_{x\in\mathcal{X}}],\shuffle,1_{\mathcal{X}^*})$ *and* $(\mathbb{C}\langle\mathcal{X}\rangle,\shuffle,1_{\mathcal{X}^*})$ *are* $\mathbb{C}$*-algebraically disjoint and* $\{x^*,l\}_{\substack{x\in\mathcal{X}\\ l\in\mathcal{L}yn\mathcal{X}}}$ *generates freely*
$$\begin{aligned}(\mathbb{C}[\{x^*\}_{x\in\mathcal{X}}]\langle\mathcal{X}\rangle,\shuffle,1_{\mathcal{X}^*}) &\cong (\mathbb{C}[\{x^*\}_{x\in\mathcal{X}}][\mathcal{L}yn\mathcal{X}],\shuffle,1_{\mathcal{X}^*})\\ &\cong (\mathbb{C}[\{x^*,l\}_{\substack{x\in\mathcal{X}\\ l\in\mathcal{L}yn\mathcal{X}}}],\shuffle,1_{\mathcal{X}^*}).\end{aligned}$$

2. *Let* $\varphi:(\mathbb{C}[\{x^*\}_{x\in\mathcal{X}}]\langle\mathcal{X}\rangle,\shuffle,1_{\mathcal{X}^*})\to(\mathbb{C},\times,1)$ *be a* $\shuffle$*-morphism. Let* $K:=\mathbb{C}[\{\varphi(x^*)\}_{x\in\mathcal{X}}]$ *and* $F:=\mathbb{C}[\{\varphi(l)\}_{l\in\mathcal{L}yn\mathcal{X}}]$.

   Then the following assertions are equivalent

   2.1 *The morphism* φ *is injective.*

   2.2 *The algebras* K *and* F, *satisfying* $K\cap F=\mathbb{C}.1$, *are generated by the transcendent bases* $\{\varphi(x^*)\}_{x\in\mathcal{X}}$ *and* $\{\varphi(l)\}_{l\in\mathcal{L}yn\mathcal{X}}$, *respectively, over* $\mathbb{C}$.

   If 1, or 2, holds then F *and* K *are* $\mathbb{C}$*-algebraically disjoint and* $\{\varphi(x^*),\varphi(l)\}_{\substack{x\in\mathcal{X}\\ l\in\mathcal{L}yn\mathcal{X}}}$ *generates freely* $\mathbb{C}[\{\varphi(x^*)\}_{x\in\mathcal{X}}][\{\varphi(l)\}_{l\in\mathcal{L}yn\mathcal{X}}]\cong\mathbb{C}[\{\varphi(x^*),\varphi(l)\}_{\substack{x\in\mathcal{X}\\ l\in\mathcal{L}yn\mathcal{X}}}]$.

MORPHISMS $\begin{matrix}\mathrm{Li}_\bullet\\ \mathrm{H}_\bullet\end{matrix}$ AND $\begin{matrix}\zeta_{\sqcup\!\sqcup}\\ \gamma_\bullet\end{matrix}$ OF $\begin{matrix}(\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle, ш, 1_{X^*})\\ (\mathbb{C}^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \boxplus\!\!\!\sqcup, 1_{Y^*})\end{matrix}$

Iterated integrals and representative series

$(\mathcal{H}(\Omega), 1_\Omega)$: the ring of hol. funct. on the simply connected dom. Ω of $\mathbb{C}$. The iterated integral, over $\{\omega_i\}_{i\geq 1}$ and along $z_0 \rightsquigarrow z$ on Ω, is defined by

$$\alpha_{z_0}^{z}(1_{\mathcal{X}^*}) = 1_\Omega, \quad \text{and} \quad \alpha_{z_0}^{z}(x_{i_1}\dots x_{i_k}) = \int_{z_0}^{z} \omega_{i_1}(z_1)\dots \int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k),$$

satisfying $\alpha_{z_0}^{z}(w \sqcup\!\sqcup v) = \alpha_{z_0}^{z}(w)\alpha_{z_0}^{z}(v)$ ($w, v \in \mathcal{X}^*$, Chen's lemma). Hence, $\forall x \in \mathcal{X}, k \geq 0, \alpha_{z_0}^{z}(x^k) = (\alpha_{z_0}^{z}(x))^k/k!$ and then $\alpha_{z_0}^{z}(x^*) = e^{\alpha_{z_0}^{z}(x)}$.

Example (with $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$)

$$\alpha_1^z(x_0^k) = \int_1^z \omega_0(z_1)\dots \int_1^{z_{k-1}} \omega_0(z_{k-1}) = \frac{\log^k(z)}{k!}.$$

$$\alpha_0^z(x_1^k) = \int_0^z \omega_1(z_1)\dots \int_0^{z_{k-1}} \omega_1(z_{k-1}) = \mathrm{Li}_{1,\dots,1}(z) = \frac{\log^k((1-z)^{-1})}{k!}.$$

$$\alpha_0^z(x_0x_1) = \int_0^z \frac{ds}{s}\int_0^s \frac{dt}{1-t} = \int_0^z \frac{ds}{s}\int_0^s dt \sum_{k\geq 0} t^k = \sum_{k\geq 1}\int_0^z ds\frac{s^{k-1}}{k} = \sum_{k\geq 1}\frac{z^k}{k^2} = \mathrm{Li}_2(z).$$

$$\alpha_0^z(x_0^{s_1-1}x_1\dots x_0^{s_k-1}x_1) = \mathrm{Li}_{s_1,\dots,s_r}(z) =: \mathrm{Li}_{x_0^{s_1-1}x_1\dots x_0^{s_k-1}x_1}(z).$$

Example (with $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$)

$$\alpha_1^z(x_0^*) = z, \quad \alpha_1^z((-x_0)^*) = z^{-1}, \quad \alpha_0^z(x_1^*) = (1-z)^{-1}.$$

$$\mathrm{Li}_{x_0^*}(z) = z, \quad \mathrm{Li}_{(-x_0)^*}(z) = z^{-1}, \quad \mathrm{Li}_{x_1^*}(z) = (1-z)^{-1}.$$

Isomorphism of algebras of polynomials $\mathrm{Li}_\bullet$

1. The following morphism of algebras is injective

$$\begin{array}{rcl} \mathrm{Li}_\bullet : (\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \longrightarrow & (\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}, ., 1), \\ x_0 & \longmapsto & \log(z) \\ x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1 & \longmapsto & \mathrm{Li}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1} = \mathrm{Li}_{s_1,\ldots,s_r}. \end{array}$$

$\{\mathrm{Li}_w\}_{w\in X^*}$ is $\mathbb{C}$-lin. indpt and then $\{\mathrm{Li}_{S_l}\}_{l\in\mathcal{L}ynX}$ is $\mathbb{C}$-alg. indpt.
One defines then

$$\mathrm{L} := \sum_{w\in X^*} \mathrm{Li}_w\, w = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{S_l} P_l} \quad \text{and}^{19} \quad Z_ш := \prod_{l\in\mathcal{L}ynX\setminus X}^{\searrow} e^{\mathrm{Li}_{S_l}(1) P_l}.$$

$\mathrm{L}(z) \sim_0 e^{x_0\log(z)}$ and $\mathrm{L}(z) \sim_1 e^{-x_1\log(1-z)} Z_ш$ and L satisfies[20]

$$\mathbf{d}S = (\omega_0 x_0 + \omega_1 x_1)S \quad \text{s.t.} \quad \sum_{w\in X^*} \alpha_{z_0}^z(w) w =: C_{z_0\rightsquigarrow z}.$$

2. The following morphism of algebras is injective

$$\begin{array}{rcl} \mathrm{P}_\bullet : (\mathbb{C}\langle Y\rangle, ⧢, 1_{Y^*}) & \longrightarrow & (\mathbb{C}\{\mathrm{P}_w\}_{w\in Y^*}, \odot, 1), \\ w & \longmapsto & \mathrm{P}_w(z) := \dfrac{\mathrm{Li}_{\pi_X w}(z)}{1-z} = \sum_{n\geq 0} \mathrm{H}_w(n) z^n. \end{array}$$

$\{\mathrm{P}_w\}_{w\in Y^*}$ is $\mathbb{C}$-lin. indpt and then $\{\mathrm{P}_{\Sigma_l}\}_{l\in\mathcal{L}ynY}$ is $\mathbb{C}$-alg. indpt.

---

[19] $\forall l \in \mathcal{L}ynX \setminus X, S_l$ is polynomial on $\lambda \in \mathcal{L}ynX \setminus X \subset x_0X^*x_1$.

[20] $\mathrm{L}(z) = C_{z_0\rightsquigarrow z}\mathrm{L}^{-1}(z_0)$ and, for $z_0 \to 0$, $\mathrm{L}(z)$ normalizes $C_{z_0\rightsquigarrow z}$ and $\mathrm{L}(z_0)$ is a counter term.

Isomorphism of algebras of polynomials $\mathrm{H}_\bullet$

1. The following morphism of algebras is injective[21]

$$\begin{array}{rcl}\mathrm{H}_\bullet : (\mathbb{C}\langle Y\rangle, \,⧢\,, 1_{Y^*}) & \longrightarrow & (\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*}, ., 1),\\ y_{s_1}\ldots y_{s_r} & \longmapsto & \mathrm{H}_{y_{s_1}\ldots y_{s_r}} = \mathrm{H}_{s_1,\ldots,s_r}.\end{array}$$

$\{\mathrm{H}_w\}_{w\in Y^*}$ is $\mathbb{C}$-lin. indpt and then $\{\mathrm{H}_{\Sigma_l}\}_{l\in\mathcal{L}ynY}$ is $\mathbb{C}$-alg. indpt.
One defines then

$$\mathrm{H} := \sum_{w\in Y^*}\mathrm{H}_w w = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\mathrm{H}_{\Sigma_l}(n)\Pi_l} \quad\text{and}\quad Z_{⧢} := \prod_{l\in\mathcal{L}ynY\setminus\{y_1\}}^{\searrow} e^{\mathrm{H}_{\Sigma_l}(+\infty)\Pi_l}$$

Let $\mathtt{Const} := \exp\left(-\sum_{k\geq 0}\mathrm{H}_{y_k}\frac{(-y_1)^k}{k}\right)$. Then $\mathrm{H}(n) \sim_{+\infty} \mathtt{Const(n)}\pi_Y Z_{ш}$.

2. $\forall s_1,\ldots,s_r\in\mathbb{N}, \exists\alpha_j,\eta_i\in\mathbb{Z},\kappa_i,\beta_j\in\mathbb{N}, c_j\in\mathcal{Z}$ and $b_i\in\mathcal{Z}'$ s.t.[22]

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) \underset{z\to 1}{\sim} \sum_{j=0}^{+\infty} c_j(1-z)^{\alpha_j}\log^{\beta_j}(1-z),$$

$$\mathrm{H}_{s_1,\ldots,s_r}(n) \underset{N\to+\infty}{\sim} \sum_{i=0}^{+\infty} b_i n^{\eta_i}\log^{\kappa_i}(n),$$

where $\mathcal{Z}'$ denotes the $\mathbb{Q}$-algebra generated by $\mathcal{Z}$ and by γ.

[21] For any $w\in Y^*, \mathrm{H}_w : \mathbb{N}\to\mathbb{Z}$.

[22] These coefficients of asymptotic expansion depend on comparison scale.

$⧢$ -character $\gamma_\bullet$

$$\forall w \in Y^*, \quad \gamma_w := \underset{n\to+\infty}{\mathrm{f.p.}}\ \mathrm{H}_w(n), \quad \{n^a \log^b(n)\}_{a\in\mathbb{Z}_{\le -1}, b\in\mathbb{N}}.$$

$\gamma_\bullet$ realizes a $⧢$-character: $\gamma_{y_1} = \gamma$ and $\forall l \in \mathcal{L}ynY \setminus \{y_1\}, \gamma_l = \mathrm{H}_l(+\infty)$.
One defines then

$$Z_\gamma := \sum_{w\in Y^*} \gamma_w w = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\gamma_{\Sigma_l}\Pi_l} = e^{\gamma y_1} Z_{⧢}.$$

In particular, $\langle Z_\gamma|\Sigma_{y_1}\rangle = \gamma$ and $\langle Z_\gamma|\Sigma_l\rangle = \gamma_{\Sigma_l} = \zeta(\Sigma_l)$, for $l \in \mathcal{L}ynY \setminus \{y_1\}$.

Example (convergent cases)

$$\begin{aligned}
\mathrm{Li}_{2,1}(z) &= \zeta(3) + (1-z)\log(1-z) - 1 - \tfrac{1}{2}(1-z)\log^2(1-z)\\
&+ (1-z)^2\big(-\tfrac{1}{4}\log^2(1-z) + \tfrac{1}{4}\log(1-z)\big) + \ldots,\\
\mathrm{H}_{2,1}(n) &= \zeta(3) - \tfrac{1}{n}(\log(n) + 1 + \gamma) + \tfrac{1}{2n}\log(n) + \ldots.
\end{aligned}$$

Example (divergent cases)

$$\begin{aligned}
\mathrm{Li}_{1,2}(z) &= 2 - 2\zeta(3) + \zeta(2)\log\tfrac{1}{1-z} + 2(1-z)\log\tfrac{1}{1-z}\\
&+ (1-z)\log^2\tfrac{1}{1-z} + \tfrac{1}{2}(1-z)^2\big(\log^2(1-z) - \log(1-z)\big) + \ldots,\\
\mathrm{H}_{1,2}(n) &= \zeta(2)\gamma - 2\zeta(3) + \zeta(2)\log(n) + \tfrac{1}{2n}(\zeta(2)+2) + \ldots.
\end{aligned}$$

Since $\zeta(2)\gamma = .9494817111149815245455641022317049336400059494736 6\ldots$
then $\underset{z\to1}{\mathrm{f.p.}}\,\mathrm{Li}_{1,2}(z) = 2 - 2\zeta(3) \neq \underset{n\to+\infty}{\mathrm{f.p.}}\ \mathrm{H}_{1,2}(n) = \zeta(2)\gamma - 2\zeta(3)$.

Polymorphism of algebras of polynomials ζ

The following polymorphism of algebras is surjective

$$\zeta : \frac{(\mathbb{Q}\oplus x_0\mathbb{Q}\langle X\rangle x_1, ⧢, 1_{X^*})}{(\mathbb{Q}\oplus(\mathbb{Q}\langle Y\rangle(Y\setminus\{y_1\}Y^*), \text{⧺}, 1_{Y^*})} \longrightarrow\!\!\!\!\rightarrow (\mathcal{Z}, \times, 1),$$

$$\frac{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1}{y_{s_1}\ldots y_{s_r}} \longmapsto \sum_{n_1>\ldots>n_r>0} n_1^{-s_1}\ldots n_r^{-s_r}.$$

Similar to $\gamma_\bullet$, the polymorphism ζ is extended as the following characters

$$\zeta_{⧢} : (\mathbb{Q}\langle X\rangle, ⧢, 1_{X^*}) \longrightarrow (\mathcal{Z}, \times, 1),$$
$$\zeta_{\text{⧺}} : (\mathbb{Q}\langle Y\rangle, \text{⧺}, 1_{Y^*}) \longrightarrow (\mathcal{Z}, \times, 1),$$

s.t.[23] $\zeta_{⧢}(S_{x_0}) = \log(1) = 0$ and, for any $\mathcal{L}ynX \setminus \{x_0\}$ or $l \in \mathcal{L}ynY$,

$$\zeta_{⧢}(S_l) = \langle Z_{⧢}|S_l\rangle = \underset{z\to 1}{\text{f.p.}}\,\mathrm{Li}_{S_l}(z), \quad \{(1-z)^a\log^b(1-z)\}_{a\in\mathbb{Z}_{\le -1}, b\in\mathbb{N}},$$

$$\zeta_{\text{⧺}}(\Sigma_l) = \langle Z_{\text{⧺}}|\Sigma_l\rangle = \underset{n\to+\infty}{\text{f.p.}}\,\mathrm{H}_{\Sigma_l}(n), \quad \{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}_{\le -1}, b\in\mathbb{N}}.$$

In particular, one obtains simultaneously $\zeta_{⧢}(x_1) = \zeta_{\text{⧺}}(y_1) = 0$.

$$\zeta(-s_1,\ldots,-s_r) \leftrightarrow \sum_{n_1>\ldots>n_r>0} n_1^{s_1}\ldots n_r^{s_r}, \quad \text{where } (s_1,\ldots,s_r)\in\mathbb{N}^r?$$

$$\mathrm{Li}_{-s_1,\ldots,-s_r}(z) = \sum_{k=0}^{s_1+\ldots+s_r+r} p_k(1-z)^{-k} \quad \in \mathbb{Z}[(1-z)^{-1}]$$

$$\iff \mathrm{H}_{-s_1,\ldots,-s_r}(n) = \sum_{k=0}^{s_1+\ldots+s_r+r} p_k\binom{n+k}{k} \quad \in \mathbb{Q}[n].$$

---

[23]Recall that $S_{x_0} = x_0$, $S_{x_1} = x_1$, $\Sigma_{y_1} = y_1$.

Extensions of $\mathrm{Li}_\bullet$, $\mathrm{H}_\bullet$ over $(\mathbb{C}\langle X\rangle \sqcup\!\sqcup \bigsqcup\!\sqcup_{x\in X} \mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle, \sqcup\!\sqcup, 1_{X^*})$, $(\mathbb{C}\langle Y\rangle \uplus \biguplus_{y\in Y \text{ finite}} \mathbb{C}^{\mathrm{rat}}\langle\langle y\rangle\rangle, \uplus, 1_{Y^*})$

1. By the identities $(x^*)^{\sqcup\!\sqcup\, n} = (nx)^*, (ax)^{*n} = (ax)^* \sqcup\!\sqcup (1-ax)^{n-1}$ and since $\alpha_1^z((ax_0)^*) = z^a, \alpha_0^z((bx_1)^*) = (1-z)^{-b}$ then $(n \in \mathbb{N}, x \in X, a, b \in \mathbb{C})$

$$\begin{aligned}
\mathrm{Li}_{(x_0^*)^{\sqcup\!\sqcup\, n} \sqcup\!\sqcup (x_1^*)^{\sqcup\!\sqcup\, k}}(z) &= z^n(1-z)^{-k},\\
\mathrm{Li}_{(ax_0)^{*n}}(z) &= z^a \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(a\log z)^k}{k!},\\
\mathrm{Li}_{(ax_1)^{*n}}(z) &= (1-z)^{-a} \sum_{k\geq 1}^{n-1} \binom{n-1}{k} \frac{(-a\log(1-z))^k}{k!}.
\end{aligned}$$

In particular, $\mathrm{Li}_{(x_0^*)^{\sqcup\!\sqcup\, n}}(z) = z^n$ and $\mathrm{Li}_{(x_1^*)^{\sqcup\!\sqcup\, k}}(z) = (1-z)^{-k}$.
Hence,

$$\mathrm{Li}_{x_0^*}(z) = z, \quad \mathrm{Li}_{x_1^*}(z) = (1-z)^{-1}, \quad \mathrm{Li}_{(x_0+x_1)^*}(z) = z(1-z)^{-1}.$$

2. By Newton-Girard identity, for any $y_k \in Y$ and $t \in \mathbb{C}, |\, t \,| < 1$, one has

$$\mathrm{H}_{(t^k y_k)^*} = \sum_{n\geq 0} \mathrm{H}_{y_n} t^{kn} = \exp\Big(\sum_{n\geq 1} \mathrm{H}_{y_{kn}} \frac{(-t^k)^{n-1}}{n}\Big).$$

$\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$

Proposition

$$\mathcal{C} := \mathbb{C}[z, z^{-1}, (1-z)^{-1}], \quad \mathcal{C}_0 := \mathbb{C}[z, z^{-1}], \quad \mathcal{C}_1 := \mathbb{C}[z, (1-z)^{-1}],$$
$$\mathcal{C}' := \mathbb{C}[z^{-1}, (1-z)^{-1}], \quad \mathcal{C}'_0 := \mathbb{C}[z^{-1}], \quad \mathcal{C}'_1 := \mathbb{C}[(1-z)^{-1}].$$

Let us consider the following morphisms of algebras

$$\varphi : (\mathbb{C}[x_0^*, (-x_0)^*, x_1^*], ⧢, 1_{X^*}) \to (\mathcal{C}, \times, 1_\Omega), \quad R \mapsto \mathrm{Li}_R,$$
$$\varphi' : (\mathbb{C}[x_0^*, x_1^*], ⧢, 1_{X^*}) \to (\mathcal{C}', \times, 1_\Omega), \quad R \mapsto \mathrm{Li}_R,$$
$$\varphi'_i : (\mathbb{C}[x_i^*], ⧢, 1_{X^*}) \to (\mathcal{C}'_i, \times, 1_\Omega), \quad R \mapsto \mathrm{Li}_R, i = 0, 1.$$

Let $\mathcal{G}$ be the group generated by $\{z \mapsto 1-z, z \mapsto 1/z\}$. Then

1. φ *is surjective. The shuffle-ideal* $\ker \varphi = \mathrm{span}_{\mathbb{C}}\{x_0^* ⧢ x_1^* - x_1^* + 1\}$.
2. $\varphi', \varphi'_0, \varphi'_1$ *are bijective.*
3. *For any $G \in \mathcal{C}$ and $g \in \mathcal{G}$, one has $G(g) \in \mathcal{C}$. Moreover, if $G(z) = p_1(z) + p_2(z^{-1}) + p_3((1-z)^{-1}) \in \mathcal{C}$, with $p_1, p_2, p_3 \in \mathbb{C}[z]$ s.t. $p_2(0) = p_3(0) = 0$ and $p_2, p_3 \neq 0$, then*
$$G(z) \sim_0 G_0(z) = p_2(z^{-1}) \text{ and } G(z) \sim_1 G_1(z) = p_3((1-z)^{-1}).$$
4. $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*} \cong \mathcal{C} \otimes_{\mathbb{C}} \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}$ *which is closed by $\{\theta_0, \theta_1, \iota_0, \iota_1\}$, where $\theta_0 = z\partial_z$ and $\theta_1 = (1-z)\partial_z$ and $\theta_0\iota_0 = \theta_1\iota_1 = \mathrm{Id}$. Moreover, $\ell(g(z)) \in \mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$, for $\ell \in \mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ and $g \in \mathcal{G}$.*

Extensions of $\zeta_{ш}$, $\gamma_\bullet$ over $(\mathbb{C}\langle X\rangle \sqcup\!\sqcup \bigsqcup\!\sqcup_{x\in X} \mathbb{C}^{\mathrm{rat}}\langle\!\langle x\rangle\!\rangle, ш, 1_{X^*})$ $(\mathbb{C}\langle Y\rangle \uplus \biguplus_{y\in Y,\ \text{finite}} \mathbb{C}^{\mathrm{rat}}\langle\!\langle y\rangle\!\rangle, \uplus, 1_{Y^*})$

Proposition ($\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$)

For any $y_{s_1}\ldots y_{s_r} \in (Y\cup\{y_0\})^*$ associated to $(s_1,\ldots,s_r)\in\mathbb{N}^r$, let $R_{y_{s_1}\ldots y_{s_r}} \in (\mathbb{Z}[x_1^*], ш, 1_{X^*})$ be defined by

$$R_{y_{s_1}\ldots y_{s_r}} = \sum_{k_1=0}^{s_1}\ldots\sum_{k_r=0}^{(s_1+\ldots+s_r)-(k_1+\ldots+k_{r-1})}\binom{s_1}{k_1}\ldots\binom{\sum_{i=1}^{r}s_i-\sum_{i=1}^{r-1}k_i}{k_r}\rho_{k_1}\,ш\ldots ш\,\rho_{k_r},$$

$$\rho_0 = x_1^* - 1_{X^*} \quad \text{and} \quad \rho_{k_i} = x_1^*\,ш\sum_{j=1}^{k_i}S_2(k_i,j)j!(x_1^*-1_{X^*})^{ш j},$$

where the $S_2(k_i,j)$'s are Stirling numbers of second kind. Then $\alpha_0^z(R_{y_{s_1}\ldots y_{s_r}}) = \mathrm{Li}_{R_{y_{s_1}\ldots y_{s_r}}}(z) = \mathrm{Li}_{-s_1,\ldots,-s_r}(z)$, $\mathrm{H}_{\pi_Y(R_{y_{s_1}\ldots y_{s_r}})} = \mathrm{H}_{-s_1,\ldots,-s_r}$.

$\zeta_{ш}$, $\gamma_\bullet$ are extended, for any $t\in\mathbb{C}$ s.t. $|t|<1$, as follows

$$\zeta_{ш}((tx_0)^*) = \zeta_{ш}((tx_1)^*) = 1 \quad \text{and} \quad \gamma_{(ty_k)^*} = \Gamma_{y_k}^{-1}(1+t),$$

$$\text{where}^{24} \quad \Gamma_{y_k}(1+t) := \begin{cases} \exp\left(\gamma t - \sum_{n\geq 2}\zeta(n)\frac{(-t)^n}{n}\right), & \text{if} \quad k=1,\\ \exp\left(-\sum_{n\geq 1}\zeta(kn)\frac{(-t^k)^n}{n}\right), & \text{if} \quad k\geq 2.\end{cases}$$

[24] In particular, Γ_{y_1} is the eulerian Gamma function, Γ.

Computational examples

Example

$$
\begin{aligned}
\mathrm{Li}_{-1,-1}(z) &= -\mathrm{Li}_{x_1^*}(z) + 5\,\mathrm{Li}_{(2x_1)^*}(z) - 7\,\mathrm{Li}_{(3x_1)^*}(z) + 3\,\mathrm{Li}_{(4x_1)^*}(z) \\
&= -(1-z)^{-1} + 3(1-z)^{-2} + 3(1-z)^{-3} - (1-z)^{-4}, \\
\mathrm{Li}_{-2,-1}(z) &= \mathrm{Li}_{x_1^*}(z) - 11\,\mathrm{Li}_{(2x_1)^*}(z) + 31\,\mathrm{Li}_{(3x_1)^*}(z) - 33\,\mathrm{Li}_{(4x_1)^*}(z) + 12\,\mathrm{Li}_{(5x_1)^*}(z) \\
&= (1-z)^{-1} - 9(1-z)^{-2} + 17(1-z)^{-3} - 23(1-z)^{-4} - 14(1-z)^{-5}, \\
\mathrm{Li}_{-1,-2}(z) &= \mathrm{Li}_{x_1^*}(z) - 9\,\mathrm{Li}_{(2x_1)^*}(z) + 23\,\mathrm{Li}_{(3x_1)^*}(z) - 23\,\mathrm{Li}_{(4x_1)^*}(z) + 8\,\mathrm{Li}_{(5x_1)^*}(z) \\
&= (1-z)^{-1} - 7(1-z)^{-2} + 9(1-z)^{-3} - 13(1-z)^{-4} - 18(1-z)^{-5}.
\end{aligned}
$$

Example

$$
\begin{aligned}
\zeta_{\sqcup\!\sqcup}(-1,-1) &= 0, \\
\zeta_{\sqcup\!\sqcup}(-2,-1) &= -1, \\
\zeta_{\sqcup\!\sqcup}(-1,-2) &= 0.
\end{aligned}
$$

Example

$$
\begin{aligned}
\mathrm{H}_{-1,-1}(n) &= -\mathrm{H}_{y_1^*}(n) + 5\mathrm{H}_{(2y_1)^*}(n) - 7\mathrm{H}_{(3y_1)^*}(n) + 3\mathrm{H}_{(4y_1)^*}(n) \\
&= n(n-1)(3n+2)(n+1)/24, \\
\mathrm{H}_{-2,-1}(n) &= \mathrm{H}_{y_1^*}(n) - 11\mathrm{H}_{(2y_1)^*}(n) + 31\mathrm{H}_{(3y_1)^*}(n) - 33\mathrm{H}_{(4y_1)^*}(n) + 12\mathrm{H}_{(5y_1)^*}(n) \\
&= n(n^2-1)(10n^2+15n+2)/120, \\
\mathrm{H}_{-1,-2}(n) &= \mathrm{H}_{y_1^*}(n) - 9\mathrm{H}_{(2y_1)^*}(n) + 23\mathrm{H}_{(3y_1)^*}(n) - 23\mathrm{H}_{(4y_1)^*}(n) + 8\mathrm{H}_{(5y_1^*)}(n) \\
&= n(10n^5 + 12n^4 - 10n^3 - 35n^2 + 5n + 3)/180.
\end{aligned}
$$

Example

$$
\begin{aligned}
\gamma_{-1,-1} &= -\Gamma^{-1}(2) + 5\Gamma^{-1}(3) - 7\Gamma^{-1}(4) + 3\Gamma^{-1}(5) = \frac{11}{24}, \\
\gamma_{-2,-1} &= \Gamma^{-1}(2) - 11\Gamma^{-1}(3) + 31\Gamma^{-1}(4) - 33\Gamma^{-1}(5) + 12\Gamma^{-1}(6) = -\frac{73}{120}, \\
\gamma_{-1,-2} &= \Gamma^{-1}(2) - 9\Gamma^{-1}(3) + 23\Gamma^{-1}(4) - 23\Gamma^{-1}(5) + 8\Gamma^{-1}(6) = -\frac{67}{120}.
\end{aligned}
$$

Example using identity $(ty_1)^* ⧢ (-ty_1)^* = (-t^2y_2)^*$

Since $\gamma_{(-t^2y_2)^*} = \Gamma_{y_2}^{-1}(1+\mathrm{i}t)$, $\gamma_{(ty_1)^*} = \Gamma_{y_1}^{-1}(1+t)$, $\gamma_{(-ty_1)^*} = \Gamma_{y_1}^{-1}(1+t)$
then $\gamma_{(-t^2y_2)^*} = \gamma_{(ty_1)^*}\gamma_{(-ty_1)^*}$, *i.e.* $\Gamma_{y_2}^{-1}(1-t) = \Gamma_{y_1}^{-1}(1+t)\Gamma_{y_1}^{-1}(1-t)$.
By the definitions of $\Gamma_{y_1}, \Gamma_{y_2}$ and the Euler's complement formula, one gets

$$\exp\left(-\sum_{k\geq 2}\zeta(2k)\frac{t^{2k}}{k}\right) = \frac{\sin(t\pi)}{t\pi} = \sum_{k\geq 1}\frac{(t\mathrm{i}\pi)^{2k}}{(2k)!}.$$

Hence, $$-\sum_{k\geq 1}\zeta(2k)\frac{u^{2k}}{k} = \sum_{k\geq 1}(u\mathrm{i}\pi)^{2k}\sum_{l\geq 1}\frac{(-1)^{l-1}}{l}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=k}}\prod_{i=1}^{l}\frac{1}{\Gamma(2n_i+2)}.$$

One can deduce then the following expression for $\zeta(2k)$:

$$\frac{\zeta(2k)}{\pi^{2k}} = k\sum_{l=1}^{k}\frac{(-1)^{k+l}}{l}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=k}}\prod_{i=1}^{l}\frac{1}{\Gamma(2n_i+2)} \in \mathbb{Q}.$$

Example

$$\begin{aligned}
\frac{\zeta(2)}{\pi^2} &= \frac{1}{\Gamma(4)} &&= \frac{1}{6},\\
\frac{\zeta(4)}{\pi^4} &= 2\left(\frac{(-1)^{2+1}}{1}\frac{1}{\Gamma(6)} + \frac{(-1)^{2+2}}{2}\frac{1}{\Gamma(4)\Gamma(4)}\right) &&= \frac{1}{90},\\
\frac{\zeta(6)}{\pi^6} &= 3\sum_{l=1}^{3}\frac{(-1)^{3+l}}{l}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=3}}\prod_{i=1}^{l}\frac{1}{\Gamma(2n_i+2)} &&= \frac{1}{945},\\
\frac{\zeta(8)}{\pi^8} &= 4\sum_{l=1}^{4}\frac{(-1)^{4+l}}{l}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=4}}\prod_{i=1}^{l}\frac{1}{\Gamma(2n_i+2)} &&= \frac{1}{9450},\\
\frac{\zeta(10)}{\pi^{10}} &= 5\sum_{l=1}^{5}\frac{(-1)^{5+l}}{l}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=5}}\prod_{i=1}^{l}\frac{1}{\Gamma(2n_i+2)} &&= \frac{1}{93555}.
\end{aligned}$$

Example using identities

$$\begin{array}{rcl}(-t^2y_2)^* \sqcup\!\!\sqcup (t^2y_2)^* & = & (-t^4y_4)^* \\ (-t^2x_0x_1)^* \sqcup\!\!\sqcup (t^2x_0x_1)^* & = & (-4t^4x_0^2x_1^2)^*\end{array}$$

In the previous example, since $\gamma_{(-t^2y_2)^*} = \zeta((-t^2y_2)^*)$ then

$$\sum_{k\geq 0} \zeta(\overbrace{2,\ldots,2}^{k\text{times}})(-1)^k t^{2k} = \frac{\sin(t\pi)}{t\pi} = \sum_{k\geq 1}(-1)^k \frac{(t\pi)^{2k}}{(2k+1)!}.$$

Then, identifying the coefficients of t^{2k}, one obtains

$$\frac{\zeta(\overbrace{2,\ldots,2}^{k\text{times}})}{\pi^{2k}} = \frac{1}{(2k+1)!} \in \mathbb{Q}.$$

Since $\gamma_{(-t^4y_4)^*} = \Gamma_{y_4}^{-1}(1-t), \gamma_{(-t^2y_2)^*} = \Gamma_{y_2}^{-1}(1-t), \gamma_{(t^2y_2)^*} = \Gamma_{y_2}^{-1}(1+t)$ then $\gamma_{(-t^4y_4)^*} = \gamma_{(t^2y_2)^*}\gamma_{(-t^2y_2)^*}$ *i.e.* $\Gamma_{y_4}^{-1}(1-t) = \Gamma_{y_2}^{-1}(1+t)\Gamma_{y_2}^{-1}(1-t)$.
By the definitions of $\Gamma_{y_2}, \Gamma_{y_4}$, one gets (see also previous example)

$$\exp\Big(-\sum_{k\geq 1}\zeta(4k)\frac{t^{4k}}{k}\Big) = \frac{\sin(\mathrm{i}t\pi)}{\mathrm{i}t\pi}\frac{\sin(t\pi)}{t\pi} = \sum_{k\geq 1}\frac{2(-4t\pi)^{4k}}{(4k+2)!}.$$

$\gamma_{(-t^4y_4)^*} = \zeta((-t^4y_4)^*), \gamma_{(-t^2y_2)^*} = \zeta((-t^2y_2)^*), \gamma_{(t^2y_2)^*} = \zeta((t^2y_2)^*)$. Then

$$\begin{array}{rcll}\zeta((-t^4y_4)^*) & = & \zeta((-t^2y_2)^*)\zeta((t^2y_2)^*) & = \zeta((-t^2x_0x_1)^*)\zeta((t^2x_0x_1)^*)) \\ & = & \zeta((-t^2x_0x_1)^* \sqcup\!\!\sqcup (t^2x_0x_1)^*) & = \zeta((-4t^4x_0^2x_1^2)^*).\end{array}$$

Expanding the Kleene stars and identifying the coefficients of t^{4k}, one gets

$$\frac{\zeta(\overbrace{3,1,\ldots,3,1}^{k\text{times}})}{\pi^{4k}} = \frac{\zeta(\overbrace{4,\ldots,4}^{k\text{times}})}{4^k\pi^{4k}} = \frac{2}{(4k+2)!} \in \mathbb{Q}.$$

APPLICATION TO ZAGIER'S DIMENSION CONJECTURE

Abel-like results obtained by Hopf-like techniques

Theorem (Abel-like theorem and bridge equations)

$$\lim_{z\to 1} e^{y_1 \log(1-z)}\pi_Y \mathrm{L}(z) = \lim_{n\to\infty} \mathtt{Const(n)}^{-1}\mathrm{H}(n) = \pi_Y Z_{\sqcup\!\sqcup}.$$

Hence, $Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\!\sqcup}$*, or equivalently by cancellation,* $Z_{\uplus} = B'(y_1)\pi_Y Z_{\sqcup\!\sqcup}$*,* *where* $B(y_1) = \exp\left(\gamma y_1 - \sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\right)$ *and* $B'(y_1) = \exp\left(\sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\right)$.

Identifying the coefficients of $y_1^k w$ in $Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\!\sqcup}$, one obtains[25]

$$\gamma_{y_1^k} = \sum_{\substack{s_1,\ldots,s_k>0\\ s_1+\ldots+ks_k=k}} \frac{(-1)^k}{s_1!\ldots s_k!}(-\gamma)^{s_1}\left(-\frac{\zeta(2)}{2}\right)^{s_2}\ldots\left(-\frac{\zeta(k)}{k}\right)^{s_k},$$

$$\gamma_{y_1^k w} = \sum_{i=0}^{k}\frac{\zeta(x_0(-x_1)^{k-i} \sqcup\!\sqcup \pi_X w])}{i!}\left(\sum_{j=1}^{i} b_{i,j}(\gamma, -\zeta(2), 2\zeta(3), \ldots)\right),$$

where the $b_{n,k}(t_1,\ldots,t_k)$'s are Bell polynomials, $k\in\mathbb{N}_+$ and $w\in Y^*Y$.

Example (Generalized Euler's gamma constant)

$$\begin{aligned}
\gamma_{1,1} &= \tfrac{1}{2}(\gamma^2 - \zeta(2)),\\
\gamma_{1,1,1} &= \tfrac{1}{6}(\gamma^3 - 3\zeta(2)\gamma + 2\zeta(3)),\\
\gamma_{1,7} &= \zeta(7)\gamma + \zeta(3)\zeta(5) - \tfrac{54}{175}\zeta(2)^4,\\
\gamma_{1,1,6} &= \tfrac{4}{35}\zeta(2)^3\gamma^2 + (\zeta(2)\zeta(5) + \tfrac{2}{5}\zeta(3)\zeta(2)^2 - 4\zeta(7))\gamma + \zeta(6,2)\\
&+ \tfrac{19}{35}\zeta(2)^4 + \tfrac{1}{2}\zeta(2)\zeta(3)^2 - 4\zeta(3)\zeta(5).
\end{aligned}$$

[25] Recall that $(s_1,\ldots,s_r) \leftrightarrow y_{s_1}\ldots y_{s_r} \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1$

Algorithm **LocaleCordinateIdentification**

$\mathcal{Z}_{irr}^{\infty}(\mathcal{X}) := \{\}, \mathcal{L}_{irr}^{\infty}(\mathcal{X}) := \{\}, \mathcal{R}_{irr}(\mathcal{X}) := \{\};$

for p range in $2, \ldots, \infty$ do

for l range in the totally ordered[26] $\mathcal{L}yn^p(\mathcal{X})$ do

identify the coefficients of Π_l in $Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\!\sqcup}$;

identify the coefficients of P_l in $\pi_X Z_\gamma = B(x_1) Z_{\sqcup\!\sqcup}$

end_for;

by elimination, obtain the system of equations in $\{\zeta(\Sigma_l)\}_{l \in \mathcal{L}yn^p(X)}$;

by elimination, obtain the system of equations in $\{\zeta(S_l)\}_{l \in \mathcal{L}yn^p(Y)}$;

for l range in the totally ordered $\mathcal{L}yn^p(\mathcal{X})$ do

express the equation led by $\zeta(\Sigma_l)$ as rewriting rule;

if $\zeta(\Sigma_l) \to \zeta(\Sigma_l)$

then $\mathcal{Z}_{irr}^{\infty}(Y) := \mathcal{Z}_{irr}^{\infty}(Y) \cup \{\zeta(\Sigma_l)\}$ and $\mathcal{L}_{irr}^{\infty}(Y) := \mathcal{L}_{irr}^{\infty}(Y) \cup \{\Sigma_l\}$

else $\mathcal{R}_{irr}(Y) := \mathcal{R}_{irr}(Y) \cup \{\Sigma_l \to \Upsilon_l\}$;

express the equation led by $\zeta(S_l)$ as rewriting rule;

if $\zeta(S_l) \to \zeta(S_l)$

then $\mathcal{Z}_{irr}^{\infty}(X) := \mathcal{Z}_{irr}^{\infty}(X) \cup \{\zeta(S_l)\}$ and $\mathcal{L}_{irr}^{\infty}(X) := \mathcal{L}_{irr}^{\infty}(X) \cup \{S_l\}$

else $\mathcal{R}_{irr}(X) := \mathcal{R}_{irr}(X) \cup \{S_l \to U_l\}$

end_for

end_for

---

[26] $\mathcal{L}yn^p(\mathcal{X})$ denotes the set of Lyndon words over $\mathcal{X}$ of weight p.

Polynomial relations on local coordinates $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX\setminus X}$ $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$

The identification of local coordinates in $Z_\gamma = B(y_1)\pi_Y Z_{⧢}$, leads to

1. A family of algebraic generators $\mathcal{Z}_{irr}^{\infty}(\mathcal{X})$ of $\mathcal{Z}$ constructed as follows
$$\mathcal{Z}_{irr}^{\leq 2}(\mathcal{X}) \subset \cdots \subset \mathcal{Z}_{irr}^{\leq k}(\mathcal{X}) \subset \cdots \subset \mathcal{Z}_{irr}^{\infty}(\mathcal{X}) = \bigcup_{k\geq 2} \mathcal{Z}_{irr}^{\leq k}(\mathcal{X})$$
and their inverse image, by a section of ζ,
$$\mathcal{L}_{irr}^{\leq 2}(\mathcal{X}) \subset \cdots \subset \mathcal{L}_{irr}^{\leq k}(\mathcal{X}) \subset \cdots \subset \mathcal{L}_{irr}^{\infty}(\mathcal{X}) = \bigcup_{k\geq 2} \mathcal{L}_{irr}^{\leq k}(\mathcal{X})$$
such that the following restriction is bijective
$$\zeta : \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})] \to \mathcal{Z} = \mathbb{Q}[\mathcal{Z}_{irr}^{\infty}(\mathcal{X})] = \mathbb{Q}[\{\zeta(p)\}_{p\in\mathcal{L}_{irr}^{\infty}(\mathcal{X})}].$$
2. A ideal $\mathcal{R}_{\mathcal{X}}$ generated by the polynomials $\{Q_l\}_{\substack{l\in\mathcal{L}yn\mathcal{X}\\ l\neq y_1,x_0,x_1}}$ homogenous in weight $(=(l))$ such that the following assertions are equivalent
   i. $Q_l = 0$,
   ii. $\Sigma_l \to \Sigma_l$ (resp. $S_l \to S_l$),
   iii. $\Sigma_l \in \mathcal{L}_{irr}^{\infty}(Y)$ (resp. $S_l \in \mathcal{L}_{irr}^{\infty}(X)$).

   $0 \neq Q_l$ is led by Σ_l (resp. S_l), being transcendent over $\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$, and $\Sigma_l \to \Upsilon_l$ (resp. $S_l \to U_l$) belonging $\mathbb{Q}[\mathcal{L}_{irr}^{\leq(l)}(\mathcal{X})]$. In other terms, $\Sigma_l = Q_l + \Upsilon_l$ (resp. $S_l = Q_l + U_l$), *i.e.*
$$\mathrm{span}_{\mathbb{Q}} \begin{matrix}\{S_l\}_{l\in\mathcal{L}ynX\setminus X}\\ \{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}\end{matrix} = \mathcal{R}_{\mathcal{X}} \oplus \mathrm{span}_{\mathbb{Q}}\mathcal{L}_{irr}^{\infty}(\mathcal{X}).$$

Homogenous polynomials relations on local coordinates

Identification local coordinates in $Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\!\sqcup}$, yields relations among $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$, or $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX\setminus X}$, which are independent from γ.

	Polynomial relations on $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$	Polynomial relations on $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX\setminus X}$
3	$\zeta(\Sigma_{y_2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) = \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) = \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) = \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) = \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) = \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) = 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) = -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) = -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) = \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) = \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) = \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) = \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) = \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) = \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) = \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) = \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) = -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) = \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) = 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) = \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) = \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) = -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) = \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) = \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) = \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) = \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$

Noetherian rewriting system & irreducible coordinates[27]

	Rewriting among		$\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$	Rewriting among		$\{\zeta(S_l)\}_{l\in\mathcal{L}ynX\setminus X}$
3	$\zeta(\Sigma_{y_2y_1})$	$\rightarrow$	$\frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2})$	$\rightarrow$	$\zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4})$	$\rightarrow$	$\frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1})$	$\rightarrow$	$\frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1})$	$\rightarrow$	$\frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2})$	$\rightarrow$	$\frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2})$	$\rightarrow$	$\frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3})$	$\rightarrow$	$\frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2})$	$\rightarrow$	$3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2})$	$\rightarrow$	$-\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1})$	$\rightarrow$	$-\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1})$	$\rightarrow$	$-\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1})$	$\rightarrow$	$\frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3})$	$\rightarrow$	$-\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2})$	$\rightarrow$	$\frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2})$	$\rightarrow$	$\frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3})$	$\rightarrow$	$\frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4})$	$\rightarrow$	$\zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6})$	$\rightarrow$	$\frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1})$	$\rightarrow$	$\frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2})$	$\rightarrow$	$\zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2})$	$\rightarrow$	$\frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1})$	$\rightarrow$	$\frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1})$	$\rightarrow$	$\frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2})$	$\rightarrow$	$-\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3})$	$\rightarrow$	$\frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1})$	$\rightarrow$	$3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2})$	$\rightarrow$	$\frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2})$	$\rightarrow$	$\frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1})$	$\rightarrow$	$-\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2})$	$\rightarrow$	$\frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4})$	$\rightarrow$	$\frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3})$	$\rightarrow$	$\frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3})$	$\rightarrow$	$\frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4})$	$\rightarrow$	$\frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5})$	$\rightarrow$	$\frac{8}{35}\zeta(S_{x_0x_1})^3$

$$\mathcal{Z}_{irr}^{\leq 2}(\mathcal{X}) \subset \cdots \mathcal{Z}_{irr}^{\leq p}(\mathcal{X}) \subset \cdots \subset \mathcal{Z}_{irr}^{\infty}(\mathcal{X}) = \cup_{p\geq 2}\mathcal{Z}_{irr}^{\leq p}(\mathcal{X}).$$

27 The set of irreducible local coordinates forms algebraic generator system for $\mathcal{Z}$.

Homogenous polynomials generating inside ker ζ

	$\{Q_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$	$\{Q_l\}_{l\in\mathcal{L}ynX\setminus X}$
3	$\zeta(\Sigma_{y_2y_1}-\frac{3}{2}\Sigma_{y_3})=0$	$\zeta(S_{x_0x_1^2}-S_{x_0^2x_1})=0$
4	$\zeta(\Sigma_{y_4}-\frac{2}{5}\Sigma_{y_2}^{\uplus 2})=0$	$\zeta(S_{x_0^3x_1}-\frac{2}{5}S_{x_0x_1}^{\sqcup\!\sqcup 2})=0$
	$\zeta(\Sigma_{y_3y_1}-\frac{3}{10}\Sigma_{y_2}^{\uplus 2})=0$	$\zeta(S_{x_0^2x_1^2}-\frac{1}{10}S_{x_0x_1}^{\sqcup\!\sqcup 2})=0$
	$\zeta(\Sigma_{y_2y_1^2}-\frac{2}{3}\Sigma_{y_2}^{\uplus 2})=0$	$\zeta(S_{x_0x_1^3}-\frac{2}{5}S_{x_0x_1}^{\sqcup\!\sqcup 2})=0$
5	$\zeta(\Sigma_{y_3y_2}-3\Sigma_{y_3}\uplus\Sigma_{y_2}-5\Sigma_{y_5})=0$	$\zeta(S_{x_0^3x_1^2}-S_{x_0^2x_1}\sqcup\!\sqcup S_{x_0x_1}+2S_{x_0^4x_1})=0$
	$\zeta(\Sigma_{y_4y_1}-\Sigma_{y_3}\uplus\Sigma_{y_2})+\frac{5}{2}\Sigma_{y_5})=0$	$\zeta(S_{x_0^2x_1x_0x_1}-\frac{3}{2}S_{x_0^4x_1}+S_{x_0^2x_1}\sqcup\!\sqcup S_{x_0x_1})=0$
	$\zeta(\Sigma_{y_2^2y_1}-\frac{3}{2}\Sigma_{y_3}\uplus\Sigma_{y_2}-\frac{25}{12}\Sigma_{y_5})=0$	$\zeta(S_{x_0^2x_1^3}-S_{x_0^2x_1}\sqcup\!\sqcup S_{x_0x_1}+2S_{x_0^4x_1})=0$
	$\zeta(\Sigma_{y_3y_1^2}-\frac{5}{12}\Sigma_{y_5})=0$	$\zeta(S_{x_0x_1x_0x_1^2}-\frac{1}{2}S_{x_0^4x_1})=0$
	$\zeta(\Sigma_{y_2y_1^3}-\frac{1}{4}\Sigma_{y_3}\uplus\Sigma_{y_2})+\frac{5}{4}\Sigma_{y_5})=0$	$\zeta(S_{x_0x_1^4}-S_{x_0^4x_1})=0$
6	$\zeta(\Sigma_{y_6}-\frac{8}{35}\Sigma_{y_2}^{\uplus 3})=0$	$\zeta(S_{x_0^5x_1}-\frac{8}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3})=0$
	$\zeta(\Sigma_{y_4y_2}-\Sigma_{y_3}^{\uplus 2}-\frac{4}{21}\Sigma_{y_2}^{\uplus 3})=0$	$\zeta(S_{x_0^4x_1^2}-\frac{6}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3}-\frac{1}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$
	$\zeta(\Sigma_{y_5y_1}-\frac{2}{7}\Sigma_{y_2}^{\uplus 3}-\frac{1}{2}\Sigma_{y_3}^{\uplus 2})=0$	$\zeta(S_{x_0^3x_1x_0x_1}-\frac{4}{105}S_{x_0x_1}^{\sqcup\!\sqcup 3})=0$
	$\zeta(\Sigma_{y_3y_1y_2}-\frac{17}{30}\Sigma_{y_2}^{\uplus 3}+\frac{9}{4}\Sigma_{y_3}^{\uplus 2})=0$	$\zeta(S_{x_0^3x_1^3}-\frac{23}{70}S_{x_0x_1}^{\sqcup\!\sqcup 3}-S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$
	$\zeta(\Sigma_{y_3y_2y_1}-3\Sigma_{y_3}^{\uplus 2}-\frac{9}{10}\Sigma_{y_2}^{\uplus 3})=0$	$\zeta(S_{x_0^2x_1x_0x_1^2}-\frac{2}{105}S_{x_0x_1}^{\sqcup\!\sqcup 3})=0$
	$\zeta(\Sigma_{y_4y_1^2}-\frac{3}{10}\Sigma_{y_2}^{\uplus 2}-\frac{3}{4}\Sigma_{y_3}^{\uplus 2})=0$	$\zeta(S_{x_0^2x_1^2x_0x_1}-\frac{89}{210}S_{x_0x_1}^{\sqcup\!\sqcup 3}+\frac{3}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$
	$\zeta(\Sigma_{y_2^2y_1^2}-\frac{11}{63}\Sigma_{y_2}^{\uplus 2}-\frac{1}{4}\Sigma_{y_3}^{\uplus 2})=0$	$\zeta(S_{x_0^2x_1^4}-\frac{6}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3}-\frac{1}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$
	$\zeta(\Sigma_{y_3y_1^3}-\frac{1}{21}\Sigma_{y_2}^{\uplus 3})=0$	$\zeta(S_{x_0x_1x_0x_1^3}-\frac{8}{21}S_{x_0x_1}^{\sqcup\!\sqcup 3}-S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$
	$\zeta(\Sigma_{y_2y_1^4}-\frac{17}{50}\Sigma_{y_2}^{\uplus 3}+\frac{3}{16}\Sigma_{y_3}^{\uplus 2})=0$	$\zeta(S_{x_0x_1^5}-\frac{8}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3})=0$

One has $\left\{\begin{array}{c}\mathcal{R}_Y:=(\mathrm{span}_{\mathbb{Q}}\{Q_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}},\uplus,1_{Y^*})\\ \mathcal{R}_X:=(\mathrm{span}_{\mathbb{Q}}\{Q_l\}_{l\in\mathcal{L}ynX\setminus X},\sqcup\!\sqcup,1_{X^*})\end{array}\right\}\subseteq\ker\zeta$.

Noetherian rewriting system & totally ordered $\mathcal{L}_{irr}^{\infty}(\mathcal{X})$

	Rewriting among		$\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$	Rewriting among		$\{S_l\}_{\mathcal{L}ynX\setminus X}$
3	$\Sigma_{y_2y_1}$	$\rightarrow$	$\frac{3}{2}\Sigma_{y_3}$	$S_{x_0x_1^2}$	$\rightarrow$	$S_{x_0^2x_1}$
4	Σ_{y_4}	$\rightarrow$	$\frac{2}{5}\Sigma_{y_2}^2$	$S_{x_0^3x_1}$	$\rightarrow$	$\frac{2}{5}S_{x_0x_1}^2$
	$\Sigma_{y_3y_1}$	$\rightarrow$	$\frac{3}{10}\Sigma_{y_2}^2$	$S_{x_0^2x_1^2}$	$\rightarrow$	$\frac{1}{10}S_{x_0x_1}^2$
	$\Sigma_{y_2y_1^2}$	$\rightarrow$	$\frac{2}{3}\Sigma_{y_2}^2$	$S_{x_0x_1^3}$	$\rightarrow$	$\frac{2}{5}S_{x_0x_1}^2$
5	$\Sigma_{y_3y_2}$	$\rightarrow$	$3\Sigma_{y_3}\Sigma_{y_2} - 5\Sigma_{y_5}$	$S_{x_0^3x_1^2}$	$\rightarrow$	$-S_{x_0^2x_1}S_{x_0x_1} + 2S_{x_0^4x_1}$
	$\Sigma_{y_4y_1}$	$\rightarrow$	$-\Sigma_{y_3}\Sigma_{y_2} + \frac{5}{2}\Sigma_{y_5}$	$S_{x_0^2x_1x_0x_1}$	$\rightarrow$	$-\frac{3}{2}S_{x_0^4x_1} + S_{x_0^2x_1}S_{x_0x_1}$
	$\Sigma_{y_2^2y_1}$	$\rightarrow$	$\frac{3}{2}\Sigma_{y_3}\Sigma_{y_2} - \frac{25}{12}\Sigma_{y_5}$	$S_{x_0^2x_1^3}$	$\rightarrow$	$-S_{x_0^2x_1}S_{x_0x_1} + 2S_{x_0^4x_1}$
	$\Sigma_{y_3y_1^2}$	$\rightarrow$	$\frac{5}{12}\Sigma_{y_5}$	$S_{x_0x_1x_0x_1^2}$	$\rightarrow$	$\frac{1}{2}S_{x_0^4x_1}$
	$\Sigma_{y_2y_1^3}$	$\rightarrow$	$\frac{1}{4}\Sigma_{y_3}\Sigma_{y_2} + \frac{5}{4}\Sigma_{y_5}$	$S_{x_0x_1^4}$	$\rightarrow$	$S_{x_0^4x_1}$
6	Σ_{y_6}	$\rightarrow$	$\frac{8}{35}\Sigma_{y_2}^3$	$S_{x_0^5x_1}$	$\rightarrow$	$\frac{8}{35}S_{x_0x_1}^3$
	$\Sigma_{y_4y_2}$	$\rightarrow$	$\Sigma_{y_3}^2 - \frac{4}{21}\Sigma_{y_2}^3$	$S_{x_0^4x_1^2}$	$\rightarrow$	$\frac{6}{35}S_{x_0x_1}^3 - \frac{1}{2}S_{x_0^2x_1}^2$
	$\Sigma_{y_5y_1}$	$\rightarrow$	$\frac{2}{7}\Sigma_{y_2}^3 - \frac{1}{2}\Sigma_{y_3}^2$	$S_{x_0^3x_1x_0x_1}$	$\rightarrow$	$\frac{4}{105}S_{x_0x_1}^3$
	$\Sigma_{y_3y_1y_2}$	$\rightarrow$	$-\frac{17}{30}\Sigma_{y_2}^3 + \frac{9}{4}\Sigma_{y_3}^2$	$S_{x_0^3x_1^3}$	$\rightarrow$	$\frac{23}{70}S_{x_0x_1}^3 - S_{x_0^2x_1}^2$
	$\Sigma_{y_3y_2y_1}$	$\rightarrow$	$3\Sigma_{y_3}^2 - \frac{9}{10}\Sigma_{y_2}^3$	$S_{x_0^2x_1x_0x_1^2}$	$\rightarrow$	$\frac{2}{105}S_{x_0x_1}^3$
	$\Sigma_{y_4y_1^2}$	$\rightarrow$	$\frac{3}{10}\Sigma_{y_2}^3 - \frac{3}{4}\Sigma_{y_3}^2$	$S_{x_0^2x_1^2x_0x_1}$	$\rightarrow$	$-\frac{89}{210}S_{x_0x_1}^3 + \frac{3}{2}S_{x_0^2x_1}^2$
	$\Sigma_{y_2^2y_1^2}$	$\rightarrow$	$\frac{11}{63}\Sigma_{y_2}^3 - \frac{1}{4}\Sigma_{y_3}^2$	$S_{x_0^2x_1^4}$	$\rightarrow$	$\frac{6}{35}S_{x_0x_1}^3 - \frac{1}{2}S_{x_0^2x_1}^2$
	$\Sigma_{y_3y_1^3}$	$\rightarrow$	$\frac{1}{21}\Sigma_{y_2}^3$	$S_{x_0x_1x_0x_1^3}$	$\rightarrow$	$\frac{8}{21}S_{x_0x_1}^3 - S_{x_0^2x_1}^2$
	$\Sigma_{y_2y_1^4}$	$\rightarrow$	$\frac{17}{50}\Sigma_{y_2}^3 + \frac{3}{16}\Sigma_{y_3}^2$	$S_{x_0x_1^5}$	$\rightarrow$	$\frac{8}{35}S_{x_0x_1}^3$

$$\mathcal{L}_{irr}^{\leq 2}(\mathcal{X}) \subset \cdots \mathcal{L}_{irr}^{\leq p}(\mathcal{X}) \subset \cdots \subset \mathcal{L}_{irr}^{\infty}(\mathcal{X}) = \cup_{p\geq 2}\mathcal{L}_{irr}^{\leq p}(\mathcal{X}).$$

$$\forall l \in \left\{ \begin{array}{c} \mathcal{L}ynY \setminus \{y_1\} \\ \mathcal{L}ynX \setminus X \end{array} \right\}, \quad \left\{ \begin{array}{c} \Sigma_l \in \mathcal{L}_{irr}^{\infty}(Y) \\ S_l \in \mathcal{L}_{irr}^{\infty}(X) \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \Sigma_l \rightarrow \Sigma_l \\ S_l \rightarrow S_l \end{array} \right\} \Leftrightarrow Q_l = 0.$$

Im and ker of $\zeta : \begin{matrix}(\mathbb{Q}[\{S_l\}_{l\in\mathcal{L}ynX\setminus x}], ⧢, 1_{X^*})\\(\mathbb{Q}[\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}], ⧢\!\!\!\!\!+\,, 1_{Y^*})\end{matrix} \twoheadrightarrow (\mathcal{Z}, \times, 1)$

Proposition

$$\mathbb{Q}[\{S_l\}_{l\in\mathcal{L}ynX\setminus x}] = \mathcal{R}_X \oplus \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(X)],$$
$$\mathbb{Q}[\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}] = \mathcal{R}_Y \oplus \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(Y)].$$

We have seen that $\mathcal{R}_{\mathcal{X}} \subseteq \ker\zeta$. Now, let $Q \in \ker\zeta$, $\langle Q|1_{\mathcal{X}^*}\rangle = 0$. Then $Q = Q_1 + Q_2$ with $Q_1 \in \mathcal{R}_{\mathcal{X}}$ and $Q_2 \in \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$. Thus, $Q \equiv_{\mathcal{R}_{\mathcal{X}}} Q_1 \in \mathcal{R}_{\mathcal{X}}$.

Corollary

$\mathbb{Q}[\{\zeta(p)\}_{p\in\mathcal{L}_{irr}^{\infty}(\mathcal{X})}] = \mathcal{Z} = \mathrm{Im}\,\zeta$ *and* $\mathcal{R}_{\mathcal{X}} = \ker\zeta$.

$\mathrm{Im}\,\zeta \cong \mathbb{Q}1_{Y^*} \oplus (Y - \{y_1\})\mathbb{Q}\langle Y\rangle/\ker\zeta \cong \mathbb{Q}1_{X^*} \oplus x_0\mathbb{Q}\langle X\rangle x_1/\ker\zeta$.

Corollary

$$\mathcal{Z} = \mathbb{Q}1 \oplus \bigoplus_{k\geq 2} \mathcal{Z}_k,\ \textit{where } \mathcal{Z}_k := \mathrm{span}_{\mathbb{Q}}\{\zeta(s_1,\ldots,s_r) | \sum_{i=1}^{k} s_i = k\}_{\substack{s_1,\ldots,s_r\in\mathbb{N}_{\geq 1}\\ s_1>1, r\geq 1}}.$$

Now, let $\mathbb{Q}\langle\mathcal{X}\rangle \ni P \notin \ker\zeta$, homogenous in weight and let $\xi := \zeta(P)$. Since $\mathcal{Z}_p\mathcal{Z}_n \subset \mathcal{Z}_{p+n}$ then each monomial $\xi^n, n \geq 1$, is of different weight. Hence, ξ could not satisfy $\xi^n + a_{n-1}\xi^{n-1} + \ldots = 0$, with $a_{n-1}, \ldots \in \mathbb{Q}$. In particular, for any $s \in \mathcal{L}_{irr}^{\infty}(\mathcal{X}), s$ is homogenous in weight. Then

Corollary

For any $s \in \mathcal{L}_{irr}^{\infty}(\mathcal{X})$, $\zeta(s)$ *is transcendent over* $\mathbb{Q}$.

On the Zagier's dimension conjecture

$$\mathcal{Z}_{irr}^{\leq 12}(X) = \{\zeta(S_{x_0x_1}), \zeta(S_{x_0^2x_1}), \zeta(S_{x_0^4x_1}), \zeta(S_{x_0^6x_1}), \zeta(S_{x_0x_1^2x_0x_1^4}), \zeta(S_{x_0^8x_1}), \zeta(S_{x_0x_1^2x_0x_1^6}), \zeta(S_{x_0^{10}x_1}), \zeta(S_{x_0x_1^3x_0x_1^7}), \zeta(S_{x_0x_1^2x_0x_1^8}), \zeta(S_{x_0x_1^4x_0x_1^6})\}.$$

$$\mathcal{L}_{irr}^{\leq 12}(X) = \{S_{x_0x_1}, S_{x_0^2x_1}, S_{x_0^4x_1}, S_{x_0^6x_1}, S_{x_0x_1^2x_0x_1^4}, S_{x_0^8x_1}, S_{x_0x_1^2x_0x_1^6}, S_{x_0^{10}x_1}, S_{x_0x_1^3x_0x_1^7}, S_{x_0x_1^2x_0x_1^8}, S_{x_0x_1^4x_0x_1^6}\}.$$

$$\mathcal{Z}_{irr}^{\leq 12}(Y) = \{\zeta(\Sigma_{y_2}), \zeta(\Sigma_{y_3}), \zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_7}), \zeta(\Sigma_{y_3y_1^5}), \zeta(\Sigma_{y_9}), \zeta(\Sigma_{y_3y_1^7}), \zeta(\Sigma_{y_{11}}), \zeta(\Sigma_{y_2y_1^9}), \zeta(\Sigma_{y_3y_1^9}), \zeta(\Sigma_{y_2^2y_1^8})\}.$$

$$\mathcal{L}_{irr}^{\leq 12}(Y) = \{\Sigma_{y_2}, \Sigma_{y_3}, \Sigma_{y_5}, \Sigma_{y_7}, \Sigma_{y_3y_1^5}, \Sigma_{y_9}, \Sigma_{y_3y_1^7}, \Sigma_{y_{11}}, \Sigma_{y_2y_1^9}, \Sigma_{y_3y_1^9}, \Sigma_{y_2^2y_1^8}\}.$$

Let $d_k := \dim \mathcal{Z}_k$. Then $d_0 = 1, d_1 = 0, d_2 = 1, \quad d_k = d_{k-2} + d_{k-3}$?

Up to weight 12, the Zagier's dimension conjecture holds meaning that the elements of $\mathcal{Z}_{irr}^{\leq 12}(\mathcal{X})$ are algebraically independent over $\mathbb{Q}$.

1. For any $l \in {}_{\mathcal{L}ynX\setminus\{x_0,x_1\}}^{\mathcal{L}ynY\setminus\{y_1\}}$, one has $l \succeq {}_{x_0^{n-1}x_1}^{y_n}$ and ${}_{S_{x_0^{n-1}x_1}=x_0^{n-1}x_1}^{\Sigma_{y_n}=y_n}$,
2. $\zeta(2) = \zeta(\Sigma_{y_2}) = \zeta(S_{x_0x_1})$ is then irreducible and, by $\zeta(2k)/\pi^{2k}$, it follows that $\Sigma_{y_{2k}} = y_{2k} \notin \mathcal{L}_{irr}^{\infty}(Y)$ and $S_{x_0^{2k-1}x_1} = x_0^{2k-1}x_1 \notin \mathcal{L}_{irr}^{\infty}(X)$, for $k > 1$,
3. $\Sigma_{y_{2n+1}} = y_{2n+1} \in \mathcal{L}_{irr}^{\infty}(Y)$ and $S_{x_0^{2n}x_1} = x_0^{2n}x_1 \in \mathcal{L}_{irr}^{\infty}(X)$, for $k > 5$?

APPLICATION TO KNIZHNIK–ZAMOLODCHIKOV EQUATION

KZ_3 : simplest non-trivial case of KZ_n

Let $\mathbb{C}_*^3 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 | z_i \neq z_j, i \neq j\}$ and $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$.

$$\mathbf{d}F = \frac{1}{2\mathrm{i}\pi}\left(t_{1,2}\frac{d(z_1 - z_2)}{z_1 - z_2} + t_{1,3}\frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3}\frac{d(z_2 - z_3)}{z_2 - z_3}\right)F.$$

It can be solved, over $\mathcal{H}(\widetilde{\mathbb{C}_*^3})\langle\langle\mathcal{T}_3\rangle\rangle$. Drinfel'd proposed a following solution[28]

$$F(z_1, z_2, z_3) = (z_1 - z_2)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\mathrm{i}\pi} G\left(\frac{z_3 - z_2}{z_1 - z_2}\right),$$

where G satisfies the following noncommutative differential equation[29] on $]0, 1[$

$$(DE) \quad \mathbf{d}G = (A\omega_0 - B\omega_1)G(s), \quad \text{with } A = t_{1,2}/2\mathrm{i}\pi, B = t_{2,3}/2\mathrm{i}\pi.$$

He also stated that there is a unique solution[30] G_0 (resp. G_1) satisfying

$$G_0(s) \sim_0 e^{A\log(s)} = s^A \quad (\text{resp. } G_1(s) \sim_1 e^{-B\log(1-s)} = (1-s)^{-B})$$

and a unique series[31] Φ_{KZ}, so-called Drinfel'd series, s.t. $G_0 = G_1\Phi_{KZ}$.

He also proved that there is a group-like series[32], similar to Φ_{KZ}, with rational coefficients[33].

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[28] How to get this solution?

[29] How to integrate this equation?

[30] How to determine G_0 (resp. G_1)?

[31] How to determine Φ_{KZ}?

[32] How to determine such series?

[33] What do these coefficients represent?

Integration of KZ_3 by iteration

Let us denote $z = (z_1, z_2, z_3)$ and $s = (s_1, s_2, s_3)$, on $\widetilde{\mathbb{C}^3_*}$, and let

$$\tilde{\Omega}_2 = t_{1,3}\omega_{1,3} + t_{2,3}\omega_{2,3}, \quad \text{where} \quad \begin{cases} \omega_{1,3}(z) = (2i\pi)^{-1} d\log(z_1 - z_3), \\ \omega_{2,3}(z) = (2i\pi)^{-1} d\log(z_2 - z_3). \end{cases}$$

Let us proceed by with $V_0(z) = e^{t_{1,2}/2i\pi \log(z_1 - z_2)}$ and iteratively

$$\begin{aligned} V_l(z) &= \int_0^z e^{t_{1,2}/2i\pi(\log(z_1-z_2)-\log(s_1-s_2))}\tilde{\Omega}_2(s) V_{l-1}(s) \\ &= V_0(z) \int_0^z e^{-t_{1,2}/2i\pi \log(s_1-s_2)}\tilde{\Omega}_2(s) V_{l-1}(s). \end{aligned}$$

Then $\sum_{l\geq 0} V_l = V_0 G$, where

$$\begin{aligned} G(z) &= \sum_{m\geq 0} \sum_{t_{i_1,j_1}\ldots t_{i_m,j_m}\in\{t_{1,3},t_{2,3}\}^*} \int_0^z \omega_{i_1,j_1}(s_1)\varphi^{(0,s_1)}(t_{i_1,j_1}) \ldots \int_0^{s_{m-1}} \omega_{i_m,j_m}(s_m)\varphi^{(0,s_m)}(t_{i_m,j_m}) \\ &= \sum_{m\geq 0} \sum_{t_{i_1,j_1}\ldots t_{i_m,j_m}\in\{t_{1,3},t_{2,3}\}^*} \int_0^z \omega_{i_1,j_1}(s_1) \ldots \int_0^{s_{m-1}} \omega_{i_m,j_m}(s_m) \underbrace{\varphi^{(0,s_1)}(t_{i_1,j_1}) \ldots \varphi^{(0,s_m)}(t_{i_m,j_m})}_{=\varphi^{(0,z)}(t_{i_1,j_1}\ldots t_{i_m,j_m})}. \end{aligned}$$

and φ is defined by $\varphi^{(0,z)}(t_{i,3}) = e^{\mathrm{ad}_{-t_{1,2}/2i\pi \log(z_1-z_2)}} t_{i,3}$, for $i = 1$ or 2.

Solution of KZ_3 using generating series of polylogarithms

$$\mathbf{d}G = (\varphi(t_{1,3})\omega_{1,3} + \varphi(t_{2,3})\omega_{2,3})G.$$

In $(P_{1,2}) : z_1 - z_2 = 1$, $\varphi \equiv \mathrm{Id}$ and then, putting $(z_1, z_2, z_3) = (1, 0, s)$,

$$\mathbf{d}G = (x_1\omega_1 + x_0\omega_0)G, \quad \begin{cases} x_0 = \ \ t_{1,3}/2\mathrm{i}\pi, & \omega_0(s) = \ \ d\log(s), \\ x_1 = -t_{2,3}/2\mathrm{i}\pi, & \omega_1(s) = -d\log(1-s). \end{cases}$$

L is the solution $\mathbf{d}G = (x_1\omega_1 + x_0\omega_0)G$ satisfying asymptotic conditions:

$$\mathrm{L}(s) \sim_0 e^{x_0 \log z} \quad \text{and} \quad \mathrm{L}(s) \sim_1 e^{-x_1 \log(1-z)} Z_{ш},$$

$$\iff \lim_{z\to 0} \mathrm{L}(s)e^{-x_0 \log z} = 1 \quad \text{and} \quad \lim_{z\to 1} e^{x_1 \log(1-z)}\mathrm{L}(s) = Z_{ш},$$

and then $\Phi_{KZ} \equiv Z_{ш}$.

Let g be the hom. trans. $s \longmapsto (s - z_2)/(z_1 - z_2)$ mapping $\{z_2, z_1\}$ to $\{0, 1\}$.
$\mathrm{L}(g(s)) = \mathrm{L}((s - z_2)/(z_1 - z_2))$ is a particular solution of KZ_3 in $(P_{1,2})$.
So does[34] $\mathrm{L}((s - z_2)/(z_1 - z_2))(z_1 - z_2)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\mathrm{i}\pi}$.

Since $[t_{1,2} + t_{2,3} + t_{1,3}, t] = 0$ then $(z_1 - z_2)^{(t_{1,2}+t_{2,3}+t_{1,3})/2\mathrm{i}\pi}$ commutes with t and then with $\mathcal{H}(\widetilde{\mathbb{C}^3_*})\langle\langle\mathcal{T}_3\rangle\rangle$, for $t \in \mathcal{T}_3$. Hence, KZ_3 also admits $(z_1 - z_2)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\mathrm{i}\pi}\mathrm{L}((s - z_2)/(z_1 - z_2))$ as a particular solution in $(P_{1,2})$.

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[34] $(z_1 - z_2)^{(t_{1,2}+t_{2,3}+t_{1,3})/2\mathrm{i}\pi} = e^{((t_{1,2}+t_{2,3}+t_{1,3})/2\mathrm{i}\pi)\log(z_1-z_2)}$, being independent on $z_3 = s$ and then belonging to the differential Galois group of KZ_3.

Candidates for Drinfel'd series with rational coefficients

Let $\check{\pi}_Y$ be the morphism of algebras, defined over an algebraic basis, by

$$\forall l \in \mathcal{L}ynX - \{x_0\}, \check{\pi}_Y(S_l) = \pi_Y(S_l) \quad \text{and} \quad \check{\pi}_Y(x_0) = x_0$$

(such that $\mathrm{Li}_{R_{\check{\pi}_Y(x_0)}}(z) = \log(z)$ and then $\zeta_{⧢}(R_{\check{\pi}_Y(x_0)}) = 0$).

$$\Upsilon := \sum_{w \in Y^*} \mathrm{H}_{\check{\pi}_Y(R_w)} w \quad \text{and} \quad \Lambda := \sum_{w \in X^*} \mathrm{Li}_{R_{\check{\pi}_Y(w)}} w,$$

$$\mathbb{Q}\langle\langle Y \rangle\rangle \ni Z_\gamma^- := \sum_{w \in Y^*} \gamma_{\check{\pi}_Y(R_w)} w \quad \text{and} \quad Z_{⧢}^- := \sum_{w \in X^*} \zeta_{⧢}(R_{\check{\pi}_Y(w)}) w \in \mathbb{Z}\langle\langle X \rangle\rangle.$$

Theorem

All constant terms of $\Upsilon, \Lambda, Z_\gamma^-, Z_{⧢}^-$ equal 1 and

$$\Delta_{\uplus}(\Upsilon) = \Upsilon \otimes \Upsilon \quad \textit{and} \quad \Delta_{⧢}(\Lambda) = \Lambda \otimes \Lambda,$$

$$\Delta_{\uplus}(Z_\gamma^-) = Z_\gamma^- \otimes Z_\gamma^- \quad \textit{and} \quad \Delta_{⧢}(Z_{⧢}^-) = Z_{⧢}^- \otimes Z_{⧢}^-.$$

$$\Upsilon = \prod_{l \in \mathcal{L}ynY}^{\searrow} e^{\mathrm{H}_{\check{\pi}_Y(R_{\Sigma_l})} \Pi_l} \quad \textit{and} \quad \Lambda = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{R_{\check{\pi}_Y(S_l)}} P_l} \sim_0 e^{x_0 \log z},$$

$$Z_\gamma^- = \prod_{l \in \mathcal{L}ynY}^{\searrow} e^{\gamma_{\check{\pi}_Y(R_{\Sigma_l})} \Pi_l} \quad \textit{and} \quad Z_{⧢}^- = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\zeta_{⧢}(R_{\check{\pi}_Y(S_l)}) P_l}.$$

Moreover, for any $g \in \mathcal{G}$, there exists a morphism of linear substitution, σ_g, and a Lie series $C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X \rangle\rangle$, such that $\Lambda(g) = \sigma_g(\Lambda) e^C$.

THANK YOU FOR YOUR ATTENTION