ARITHMETIC DIFFERENTIAL EQUATIONS

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Abstract. The values of solutions of linear differential equations occasionally happen to be expressible via interesting arithmetic quantities, like the values of L-functions at integers. Several sources of this phenomenon are known, however most of the underlying identities remain unproven. In my talk I will systematically walk through examples of such identities linked with arithmetic differential equations of second order. Surprisingly enough, not all such second order instances are pullbacks of hypergeometric or Heun equations; these new arithmetic differential equations source from innocent-looking identities for π and are a subject of study in our recent work with Mark van Hoeij and Duco van Straten.

Many interesting numbers can be expressed in terms of the Euler–Gauss hypergeometric function

$$F(a,b,c \mid z) = {}_2F_1\biggl(\begin{matrix} a,\, b \\ c \end{matrix} \biggm| z\biggr) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!\,(c)_n} z^n,$$

where $(a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol (*aka* shifted factorial). At the same time, many special choices of the parameters a,b,c,z lead to interesting 'closed-form' evaluations. There are many features that make the function really special, and one of them is the (hypergeometric) linear differential equation $HF=0$, where

$$\begin{aligned} H = H(a,b,c) &= \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(z\frac{\mathrm{d}}{\mathrm{d}z}+c-1\right) - \left(z\frac{\mathrm{d}}{\mathrm{d}z}+a\right)\left(z\frac{\mathrm{d}}{\mathrm{d}z}+b\right) \\ &= z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + (c-(a+b+1)z)\frac{\mathrm{d}}{\mathrm{d}z} - ab, \end{aligned}$$

of second order, with three regular singularities, at $z=0$, 1 and ∞. Though the function possesses several algebraic-function specialisations (for example, when $b=c$), it is generically a transcendental function.

Let me first review two different methods for special evaluations in which the ${}_2F_1$-function shows up, and then connect them with a general problem of describing second-order differential equations (operators) having 'strong arithmetic flavour'.

A decade ago, Akihito Ebisu came up with a simple use of the fact that every three contiguous ${}_2F_1$-series are linked by a relation over $\mathbb{Q}(a,b,c,z)$. The term 'contiguous' refers here to the fact that the parameters a,b,c of one series differs from the corresponding parameters of the other by integers, so that Ebisu's methodology exploits the difference structure of the hypergeometric series rather than differential.

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The two however are naturally linked to each other — the recurrence equations for the coefficients of the series translate into the linear differential equation above, and this is true for more general linear differential equations.

In Ebisu's method, one fixes two contiguous series, say $F_0 = F(a,b,c \mid z)$ and $F_1 = F'(a,b,c \mid z)$ (its z-derivative) and write, for any choice of 'shift' (k,l,m) of the parameters (a,b,c),

$$F(a+k, b+l, c+m \mid z) = R_{k,l,m}(a,b,c,z)F_0(z) + Q_{k,l,m}(a,b,c,z)F_1(z),$$

where R and Q are certain computable rational functions of a,b,c,z. If we now choose an *admissible* quadruple (a,b,c,z_0), that is, such that $Q_{k,l,m}(a+kt, b+lt, c+mt, z_0) = 0$ for all $t \in \mathbb{C}$, then

$$F(a+k(t+1), b+l(t+1), c+m(t+1) \mid z_0) = R_{k,l,m}(a+kt, b+lt, c+mt, z_0)F_0(z_0).$$

At the same time the latter first-order recursion in t can be solved explicitly in terms of the gamma values; the detailed analysis of this (quite delicate!) part was recently given by Beukers and Forsgård. One example, corresponding to the shift $(k,l,m) = (2,2,1)$ and the admissible set $(a,b,c,z_0) = (0, \frac{1}{3}, \frac{5}{6}, -\frac{1}{8})$, is

$$F\big(2t, \tfrac{1}{3}+2t, \tfrac{5}{6}+t \mid -\tfrac{1}{8}\big) = \left(\frac{16}{27}\right)^t \frac{\Gamma(t+\frac{5}{6})\Gamma(\frac{2}{3})}{\Gamma(t+\frac{2}{3})\Gamma(\frac{5}{6})}.$$

In his 2017 AMS memoir "Special values of the hypergeometric series" Ebisu lists hundreds of evaluations.

One can find a certain level of similarity of Ebisu's ${}_2F_1$-evaluations with the formulae like

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2}-2t)_n(\frac{1}{2}+2t)_n(\frac{1}{4}+t)_n(\frac{3}{4}+t)_n}{n!^2(1+t)_n(\frac{1}{2}+t)_n} \left(-\frac{1}{48}\right)^n R(n,t) = \frac{16}{\pi\sqrt{3}} \, \frac{2^{8t}}{3^{2t}\binom{4t}{2t}^2},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3(\frac{1}{4}-t)_n(\frac{3}{4}-t)_n}{n!^3(1+2t)_n^2} \left(\frac{1}{16}\right)^n (120n^2+168nt+34n+20t+3) = \frac{32}{\pi^2} \, \frac{2^{8t}}{\binom{4t}{2t}^2},$$

where t is a nonnegative integer and

$$R(n,t) = \frac{56n^2+72nt+34n+16t^2+16t+3}{2n+2t+1}.$$

These are due to Jesús Guillera who uses a quite different methodology of WZ (Wilf–Zeilberger) pairs, in turn a baby version of creative telescoping. Now the hypergeometric series are more involved (because there are more Pochhammer symbols in the sums). When specialised at $t=0$ the formulae become

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{n!^3} \left(-\frac{1}{48}\right)^n (28n+3) = \frac{16}{\pi\sqrt{3}},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3(\frac{1}{4})_n(\frac{3}{4})_n}{n!^5} \left(\frac{1}{16}\right)^n (120n^2+34n+3) = \frac{32}{\pi^2}.$$

The first equality is an example of Ramanujan's formulae for $1/\pi$, which can be proven based on the modular parametrisation of the underlying ${}_3F_2$-hypergeometric series

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac14)_n(\frac34)_n}{n!^3}\big(4x(1-x)\big)^n=F(\tfrac14,\tfrac34,1\mid x)^2.$$

The result is however rigid in this case and does not lead, in any obvious way, to the t-extension. The second equality does not have such a modular interpretation, though there is some geometry and Hilbert modular form detection in our joint work with Dembélé, Panchishkin and Voight.

Let me focus from now on formulae for $1/\pi$ originating from the original work "Modular equations and approximations to π" of Ramanujan (1914). Their general shape is

$$\sum_{n=0}^{\infty}A_n(a+bn)z_0^n=\frac{\sqrt{c}}{\pi},$$

in which $f(z)=\sum_{n=0}^{\infty}A_nz^n\in\mathbb{Z}[[z]]$ and a,b,c,z_0 are nonzero rationals (more generally, $f(z)$ is a globally bounded series, $f(Cz)\in\mathcal{O}[[z]]$ for some $C\in\mathbb{Z}_{>0}$ and the ring of integers $\mathcal{O}$ of a number field, while $a,b,c,z_0\in\overline{\mathbb{Q}}^{\times}$). At the moment there are three generations of examples:

(1) *hypergeometric* ones given by Ramanujan himself, like the one we have seen and

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(3n)!}{n!^3}(1+6n)\frac{1}{6^{3n}}=\frac{3\sqrt3}{\pi},\tag{1}$$

each corresponding to a (special) ${}_3F_2$ hypergeometric $f(z)$ (which is in turn the square of a ${}_2F_1$ thanks to Clausen's identity);

(2) Ramanujan–Sato formulae, in which $f(z)$ is not any more hypergeometric but still satisfying a third order Picard–Fuchs equation and is, again, a square of a series coming from order 2 (those are always algebraic pullbacks of ${}_3F_2$ and ${}_2F_1$ hypergeometric examples, respectively); and

(3) Ramanujan-type formulae originating from Sun's conjectures (2011–present), when $f(z)$ is a solution of a *fourth* order Picard–Fuchs equation, whose (Zariski closure of the) monodromy group turns out to be an orthogonal group.

One standard example for the latter generation is

$$\sum_{n=0}^{\infty}u_nP_n\left(\frac{1+\sqrt3}{\sqrt6}\right)(7-2\sqrt3+18n)\left(\frac{2-\sqrt3}{2\sqrt6}\right)^n=\frac{27+11\sqrt3}{\pi\sqrt2};\tag{2}$$

here $\{u_n\}$ is a sequence generated by the Apéry-like recursion

$$(n+1)^2u_{n+1}-(7n^2+7n+2)u_n-8n^2u_{n-1}=0\quad\text{for } n=0,1,2,\dots,\quad u_{-1}=0,\ u_0=1,$$

while

$$P_n(y)=\sum_{m=0}^{n}\binom{n}{m}^2\left(\frac{y-1}{2}\right)^m\left(\frac{y+1}{2}\right)^{n-m}$$

are classical Legendre polynomials generated by

$$\sum_{n=0}^{\infty} P_n(y)z^n = \frac{1}{\sqrt{1-2yz+z^2}}.$$

It may not be straightforward that the sequence $\{u_n\}$ is integer-valued; one could use its different representations for that, $u_n = \sum_{k=0}^{n} \binom{n}{k}^3$ (these are known as Franel numbers) or

$$\sum_{n=0}^{\infty} u_n x^n = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} \, \frac{x^{2n}}{(1-2x)^{3n+1}},$$

the latter one shows that $\sum_{n=0}^{\infty} u_n x^n$ is a pullback of a ${}_2F_1$ hypergeometric function. The Picard–Fuchs equation for $f(z)$ behind (2) has order 4 but it decomposes as

$$\begin{aligned}\sum_{n=0}^{\infty} u_n P_n\biggl(\frac{(X+Y)(1-8XY)-14XY}{(Y-X)(1+8XY)}\biggr)\biggl(\frac{Y-X}{1+8XY}\biggr)^n \\ = (1+8XY)\biggl\{\sum_{n=0}^{\infty} u_n X^n\biggr\}\biggl\{\sum_{n=0}^{\infty} u_n Y^n\biggr\},\end{aligned}$$

an identity we showed with James Wan in 2011 (in a greater generality); this corresponds to the product of two so-called $D2$ operators. In fact, the formulae in (1) and (2) are equivalent, as they originate from the same specialisation of modular parametrisations of the underlying solutions of second order equations.

In many other formulae for $1/\pi$ from generation (3) similar decompositions were found and it was quite believed for a long time (by a narrow group of experts) that the situation is precisely like this: The function $f(z)$ is (up to an algebraic factor) a product of $f_1(z)$ and $f_2(z)$, each a pullback of a ${}_2F_1$ hypergeometric function (in every single case representing an elliptic integral). The situation is quite similar for generations (1) and (2), except there we get $f_1(z) = f_2(z)$ (Clausen-type formulae). Sun's list however contained examples where $f(z)$ is

$$\sum_{n=0}^{\infty} \binom{2n}{n} P_n(y)^2 z^n \quad \text{or} \quad \sum_{n=0}^{\infty} P_n(y)^3 z^n,$$

for which no decompositions were possible to produce. Some proofs of the formulae corresponding to the first generating function were given by me in 2013 based on its modular parametrisation for a particular choice of $z = z(y)$.

It came as a big surprise that the generating function

$$F(y,z) = \sum_{n=0}^{\infty} \binom{2n}{n} P_n(y)^2 z^n$$

as a function of two parameters y and z (when one of them is fixed, say) does not meet the expectations. It satisfies a fourth-order linear differential equation, viewed as a function of either variable, and both equations have (the Zariski closure of) the monodromy group O_4. In joint work with Mark van Hoeij and Duco van Straten we obtain the decomposition

$$F(y,z) = w\, I_+(4z,w^2)\, I_-(4z,w^2),$$

where $w = \sqrt{(1+4z)^2 - 16y^2z} + 4y\sqrt{-z}$ and

$$I_\pm(u,x) = \frac{1}{\pi}\int_0^1 \frac{1 - uv \pm v\sqrt{2u^2-2u}}{\sqrt{v(1-v)((1-v)(1-u^2v)(1+uv)^2 + x\,v(1-uv)^2)}}\,\mathrm{d}v$$

are generically *hyperelliptic* integrals. Furthermore, for each $u \in \mathbb{C}$, the function $I_\pm(u,x)$ satisfies a second-order differential equation with coefficients from $\mathcal{L}[x]$, where $\mathcal{L} = \mathbb{Q}(u, \sqrt{2u^2-2u})$. Surprisingly enough, such second-order equations (and there are infinitely many of them because of the extra parameter u), not reducible to elliptic integrals, were not recorded in the literature; they are reasonably simple counterexamples to a 1990 conjecture of Dwork (which is already disproven *finitely* many times through examples based on Shimura and Teichmüller curves defined over quadratic extensions of $\mathbb{Q}$).

Parametrisation

$$y = (x + (t^2-1)^2)\sqrt{z/x}, \quad z = \frac{1}{4(1-2t^2)}$$

rationalises the square root in the decomposition of $F(y,z)$, namely, $\sqrt{2u^2-2u} = 2tu$ and $u = 1/(1-2t^2)$, so that $\mathcal{L} = \mathbb{Q}(t)$. One of the most remarkable aspects is that the hyperelliptic integrals $I_\pm(u,x)$ satisfy second order equations, namely

$$L_\pm^{\mathrm{PF}} = L_\pm^x\big|_{x\mapsto\frac{-x}{4u^2}} \in \mathcal{L}(x)[\mathrm{d}/\mathrm{d}x].$$

This is unusual, the Picard–Fuchs equation for a hyperelliptic integral of genus g is expected to have order $2g$. For such a reduction to lower order to happen, the corresponding hyperelliptic curve and differential must have special properties. One of these, encountered in our monodromy computation, is multiplication by $\sqrt{2}$. For the underlying family of curves $C = C^{u,x} : Y^2 = H(x,u,v)$, where

$$H = v(1-v)\big((1-v)(1-u^2v)(1+uv)^2 + x\,v(1-uv)^2\big) \in \mathbb{Q}(u,x)[v],$$

we verify this property with a Humbert relation.

To add further value to our result, one should also take into account the following. After decomposing our the forth order differential operator into the product of L_+^{PF} and L_-^{PF}, we were left with solving the corresponding second order differential equations that are not pullbacks of hypergeometric (and even Heun) equations. But then there is no access to explicit formulae for their solutions! It was some tough work, massaging the differential equations via suitable changes of variables and analysing arithmetic behaviour of the solutions, to come up with explicit family of hyperelliptic integrals.

An arithmetic way of displaying our finding is via the following infinite family of Apéry-type recursions: Define degree $4n$ polynomials $u_n = u_n(t)$ by $u_n = 0$ for

$n<0$, $u_0=1$ and

$$\begin{aligned}
&(n+1)^2u_{n+1}\\
&\quad-2^2(16(t^4-6t^3-4t^2+6t-1)(n^2+n)+4t^4-24t^3-12t^2+20t-3)u_n\\
&\quad-2^{11}t(t-1)^3(t+1)(8(t^2+2t-1)n^2-2t^2-6t+3)u_{n-1}\\
&\quad+2^{18}t^2(t-1)^6(t+1)^2(2n+1)(2n-3)u_{n-2}=0 \quad \text{for } n=0,1,2,\dots .
\end{aligned}$$

Then $u_n\in\mathbb{Z}[t]$.

At the end of the day, we have realised that one can easily engineer examples starting from (essentially) any family of hyperelliptic integrals whose Jacobian possesses real multiplication. Not necessarily by a quadratic order! This gives an extremely rich structure of second order arithmetic differential equations, well beyond original expectations of Dwork.

One further remark is the Hilbert surface X_8 related to our example (and it only!) has received a differential treatment 35 years ago in the work of Takeshi Sasaki and Masaaki Yoshida. As an illustration of their general construction, they explicitly gave a rank 4 system of partial differential equations for the surface. We expect that system to be related to the one we encountered: To test this we have checked that it has a similar decomposition to rank 2, which it does.

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