Mahler measures and multiple Eisenstein values

François Brunault
UMPA – ÉNS Lyon

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Definition (Mahler, 1962)

For $P \in \mathbb{C}[x_1, \ldots, x_n] \setminus \{0\}$, define

$$m(P) = \frac{1}{(2\pi i)^n} \int_{T^n} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n}$$

where $T^n: |x_1| = \ldots = |x_n| = 1$ is the $n$-torus.
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where $T^n$: $|x_1| = \ldots = |x_n| = 1$ is the $n$-torus.

- The integral converges absolutely.
- If $P$ has coefficients in $\overline{\mathbb{Q}}$ then $m(P)$ should be a period in the sense of Kontsevich–Zagier.
- $m(P)$ measures the “size” of a polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$.
- Lehmer’s problem (1933): For $P \in \mathbb{Z}[x]$ monic irreducible, not cyclotomic, can $m(P) > 0$ be arbitrarily small?
**Theorem (Jensen, 1899)**

For $P \in \mathbb{C}[x] \setminus \{0\}$, $P = a_d \prod_{i=1}^{d} (x - \alpha_i)$, we have

$$m(P) = \log |a_d| + \sum_{k=1}^{d} \log |\alpha_k|.$$
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$$m(P) = \log |a_d| + \sum_{\substack{k=1 \atop |\alpha_k| \geq 1}}^{d} \log |\alpha_k|.$$ 

- Jensen’s formula is still useful for multivariate polynomials: it reduces an $n$-dimension integral to an $(n - 1)$-dimensional one.
- Example: using Jensen’s formula with respect to $y$, we have

$$m(1 + x + y) = \frac{1}{2\pi i} \int_{|x| = 1 \atop |1 + x| \geq 1} \log |1 + x| \frac{dx}{x} = \frac{1}{2\pi} \int_{-2\pi/3}^{2\pi/3} \log |1 + e^{i\theta}| d\theta$$

- How to evaluate further?
Timeline of identities

**Smyth (1981):** \( m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_3, 2) \)

Here \( L(\chi_3, s) = \sum_{n=1}^{\infty} \chi_3(n)/n^s \) is the Dirichlet \( L \)-function for

\[
\chi_3(n) = \begin{cases} 
1 & \text{if } n \equiv 1 \mod 3 \\
-1 & \text{if } n \equiv 2 \mod 3 \\
0 & \text{if } n \equiv 0 \mod 3
\end{cases}
\]
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\end{cases}
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The proof uses the series expansion

\[
\log|1 + e^{i\theta}| = -\text{Re} \sum_{n=1}^{\infty} \frac{e^{-in\theta}}{n}.
\]

and then integration from \( \theta = -2\pi/3 \) to \( 2\pi/3 \).
Timeline of identities

Smyth (1981): \( m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3) \)

Boyd and Deninger (1997):

\[
m\left( x + \frac{1}{x} + y + \frac{1}{y} + 1 \right) = \frac{15}{4\pi^2} L(E, 2) = L'(E, 0)
\]

where \( L(E, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \) is the \( L \)-function of the elliptic curve

\[
E : x + \frac{1}{x} + y + \frac{1}{y} + 1 = 0.
\]
Timeline of identities

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- Discovered using numerical experiments + theoretical insights.
- Proved by Rogers and Zudilin (2011).
Boyd (1998): Families of conjectural identities, such as

\[ m\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right) \overset{?}{=} c_k L'(E_k, 0) \quad (k \in \mathbb{Z}, k \neq 0, \pm 4) \]

for some rational number \( c_k \in \mathbb{Q}^\times \).
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for some rational number \( c_k \in \mathbb{Q}^\times \).

- Generalises to other families \( m(P(x, y) + k) \) where the Newton polygon of \( P(x, y) \) has \((0, 0)\) as the only interior point.
- Only finitely many such identities are proved.
- Related to the algebraic \( K \)-group \( K_2(E_k) \) and the Bloch-Beilinson regulator map \( K_2(E_k) \to \mathbb{R} \).
Conjecture (Boyd and Rodriguez Villegas, 2003):

\[ m((1 + x)(1 + y) + z) = \frac{15^2}{4\pi^4} L(E, 3) = -2L'(E, -1) \]

where \( E \) is an elliptic curve of conductor 15.
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- There are several other \( L(E, 3) \) identities, but they do not seem to come in families.
- Why does an elliptic curve appear here?
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where $E$ is an elliptic curve of conductor 15.

- There are several other $L(E, 3)$ identities, but they do not seem to come in families.
- Why does an elliptic curve appear here?
- Because

\[
E : \begin{cases} 
(1 + x)(1 + y) + z = 0 \\
(1 + \frac{1}{x})(1 + \frac{1}{y}) + \frac{1}{z} = 0.
\end{cases}
\]

- Note that $\{(1 + x)(1 + y) + z = 0\} \cap T^3 \subset E$. 
In this talk, we will consider \( L \)-functions of modular forms. If \( f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau} \) is a modular form on a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \), its \( L \)-function is defined by

\[
L(f, s) = \sum_{n=1}^{\infty} a_n n^s.
\]

Integral representation:

\[
(2\pi)^{-s} \Gamma(s) L(f, s) = \int_0^{\infty} (f(iy) - a_0) y^s dy.
\]

Meromorphic continuation to \( \mathbb{C} \) and functional equation.
In this talk, we will consider $L$-functions of modular forms. If $f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau}$ is a modular form on a congruence subgroup of $SL_2(\mathbb{Z})$, its $L$-function is defined by

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In this talk, we will consider $L$-functions of *modular forms*. If $f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau}$ is a modular form on a congruence subgroup of $\text{SL}_2(\mathbb{Z})$, its $L$-function is defined by

$$L(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

- **Integral representation:**
  $$(2\pi)^{-s} \Gamma(s) L(f, s) = \int_0^{\infty} (f(iy) - a_0) y^s \frac{dy}{y}.$$  

- **Meromorphic continuation to $\mathbb{C}$** and functional equation.
Theorem (B. 2023)

We have \( m((1 + x)(1 + y) + z) = -2L'(E, -1) \).
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- Now related to the \( K \)-group \( K_4(E) \).
- Uses joint work with Zudilin on \( K_4 \) regulators.
- Key tool: Multiple modular values

\[
\int_0^\infty f_1(iy_1)y_1^{s_1-1}dy_1 \int_{y_1}^\infty f_2(iy_2)y_2^{s_2-1} \ldots \int_{y_{n-1}}^\infty f_n(iy_n)y_n^{s_n-1}dy_n
\]

where \( f_1, \ldots, f_n \) are modular forms.
Let $P = (1 + x)(1 + y) + z$.

**Step 1: Deninger’s method**

Use Jensen’s formula with respect to $z$.

$$m(P) = \frac{1}{(2\pi i)^2} \int_{\Gamma} \eta(x, y, z)$$

where:

- $\eta$ is an explicit closed 2-form on $V_P = \{P(x, y, z) = 0\}$.
- $\Gamma = \{(x, y, z) \in V_P : |x| = |y| = 1, |z| \geq 1\}$. 
Step 2: Stokes’s theorem

In our case, the form $\eta$ happens to be exact. Writing $\eta = d\rho$, we have by Stokes’s theorem

$$m(P) = \frac{1}{(2\pi i)^2} \int_{\Gamma} d\rho = \frac{1}{(2\pi i)^2} \int_{\gamma} \rho$$

with

$$\gamma = \partial \Gamma = \{(x, y, z) \in V_P : |x| = |y| = |z| = 1\}.$$  

- $\gamma = V_P \cap T^3$ is contained in $E$.
- $\rho$ is a closed 1-form on $E$.
- So we now have a 1-dimensional integral on $E$. 
For any two functions \( f, g \) on \( E \), define

\[
\rho(f, g) = -D(f) \text{darg}(g) + \frac{1}{3} \log |g| \left( \log |1-f| \text{dlog}|f| - \log |f| \text{dlog}|1-f| \right)
\]

where \( D: \mathbb{P}^1(\mathbb{C}) \to \mathbb{R} \) is the Bloch-Wigner dilogarithm

\[
D(z) = \text{Im} \left( \sum_{n=1}^{\infty} \frac{z^n}{n^2} \right) + \log |z| \text{arg}(1 - z).
\]

**Theorem (Lalín, 2015)**

- \( \rho = \rho(-y, x) - \rho(-x, y) \).
- \( \gamma \) is a generator of \( H_1(E(\mathbb{C}), \mathbb{Z})^+ \).
Step 3: Translate in the modular world

The elliptic curve $E$ is isomorphic to the modular curve $X_1(15)$.

$$X_1(N) = \Gamma_1(N) \backslash \mathcal{H} \cup \{\text{cusps}\}$$

where

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : a, d \equiv 1 \mod N, \ c \equiv 0 \mod N \right\}.$$ 

Nice feature: the functions $x$ and $y$ on $E$ correspond to modular units on $X_1(15)$, that is, all their zeros and poles are at the cusps.

Key fact: if $u$ is a modular unit, then $d\log(u) = E_2(z)dz$ where $E_2$ is an Eisenstein series of weight 2.
We want to understand

$$\rho(u, v) = -D(u)d\text{arg}(v) + \frac{1}{3} \log |v| \left( \log |1-u| d\log |u| - \log |u| d\log |1-u| \right).$$

when $u$ and $v$ are modular units on $X_1(N)$. 
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\]
when \(u\) and \(v\) are modular units on \(X_1(N)\).

- \(\text{dlog}(u)\) and \(\text{dlog}(v)\) are Eisenstein series, so the log terms of the formula are well-understood.
We want to understand

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when \( u \) and \( v \) are modular units on \( X_1(N) \).

▸ \( \text{dlog}(u) \) and \( \text{dlog}(v) \) are Eisenstein series, so the log terms of the formula are well-understood.

▸ The challenging piece is \( D(u) \). We use

\[ d(D(u)) = \log |u| \text{darg}(1-u) - \log |1-u| \text{darg}(u) \]

▸ If \( u \) and \( 1-u \) are modular units, then \( D(u) \) is an iterated integral of Eisenstein series.
Definition

For $k \geq 1$ and $x = (x_1, x_2) \in (\mathbb{Z}/N\mathbb{Z})^2$, define the Eisenstein series

$$E_x^{(k)}(\tau) = \sum_{m,n \in \mathbb{Z}} \exp\left( \frac{2\pi i}{N} (mx_2 - nx_1) \right) \frac{(m\tau + n)^k}{(m\tau + n)^k} \in M_k(\Gamma(N))$$

For $x, y, z \in (\mathbb{Z}/N\mathbb{Z})^2$, define the multiple Eisenstein values (Manin, Brown)

$$\Lambda(x, y) := \int_0^{i\infty} E_x^{(2)}(\tau_1) d\tau_1 \int_{\tau_1}^{i\infty} E_y^{(2)}(\tau_2) d\tau_2$$

$$\Lambda(x, y, z) := \int_0^{i\infty} E_x^{(2)}(\tau_1) d\tau_1 \int_{\tau_1}^{i\infty} E_y^{(2)}(\tau_2) d\tau_2 \int_{\tau_2}^{i\infty} E_z^{(2)}(\tau_3) d\tau_3.$$  

$\rightsquigarrow$ The Mahler measure of $P$ can be written as an explicit linear combination of multiple Eisenstein values.
Theorem (B.–Zudilin, 2023)
Let \( x, y, z \in (\mathbb{Z}/N\mathbb{Z})^2 \) such that \( x + y + z = 0 \). If all the coordinates of \( x, y, z \) are non-zero, then

\[
\Re \left( \Lambda(x, y, y) - \Lambda(z, y, y) + \Lambda(y, x, x) - \Lambda(z, x, x) + \Lambda(z, y, x) + \Lambda(z, x, y) \\
- (\Lambda(y) - \Lambda(x))(\Lambda(x, y) + \Lambda(y, z) + \Lambda(z, x)) \right) = L'(F_{x,y}, -1) + c_{x,y} \zeta(3)
\]

for some explicit \( F_{x,y} \in M_2(\Gamma(N)) \), and \( c_{x,y} \in \mathbb{Q} \).
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\]

for some explicit \( F_{x,y} \in M_2(\Gamma(N)) \), and \( c_{x,y} \in \mathbb{Q} \).

Proving this formula requires two ingredients:

- **Interpolate** the multiple Eisenstein values to continuous parameters, viewing \( (\mathbb{Z}/N\mathbb{Z})^2 \) inside \( (\mathbb{R}/\mathbb{Z})^2 \) using \( (x_1, x_2) \mapsto \left( \frac{x_1}{N}, \frac{x_2}{N} \right) \).
- **Differentiate** with respect to these parameters to reduce the length of the iterated integrals.
Key lemma

For \( x = (x_1, x_2) \in (\mathbb{R}/\mathbb{Z})^2 \), we have

\[
\frac{d}{dx_2}E_x^{(2)}(\tau) = \frac{d}{d\tau}E_x^{(1)}(\tau).
\]
Key lemma

For $x = (x_1, x_2) \in (\mathbb{R}/\mathbb{Z})^2$, we have

$$\frac{d}{dx_2} E_x^{(2)}(\tau) = \frac{d}{d\tau} E_x^{(1)}(\tau).$$

So for example

$$\frac{d}{dy_2} \Lambda(x, y) = \int_0^\infty E_x^{(2)}(\tau_1) d\tau_1 \int_{\tau_1}^\infty \frac{d}{dy_2} E_y^{(2)}(\tau_2) d\tau_2$$

$$= \int_0^\infty E_x^{(2)}(\tau_1) d\tau_1 \int_{\tau_1}^\infty \frac{d}{d\tau_2} E_y^{(1)}(\tau_2) d\tau_2$$

$$= \int_0^\infty E_x^{(2)}(\tau_1)(E_y^{(1)}(\infty) - E_y^{(1)}(\tau_1)) d\tau_1.$$

This reduces a double integral to a single integral.
To prove the formula

\[
\text{Re}\left(\Lambda(x, y, y) - \Lambda(z, y, y) + \Lambda(y, x, x) - \Lambda(z, x, x) + \Lambda(z, y, x) + \Lambda(z, x, y)
\right.
\]

\[
- (\Lambda(y) - \Lambda(x))(\Lambda(x, y) + \Lambda(y, z) + \Lambda(z, x)) \right) = L'(F_{x,y},-1) + c_{x,y}\zeta(3)
\]

we differentiate the LHS with respect to $x_2$.

We get a sum of double integrals of the form

\[
\int_0^{i\infty} E^{(2)}_a(\tau_1) d\tau_1 \int_{\tau_1}^{i\infty} E^{(2)}_b(\tau_2) E^{(1)}_c(\tau_2) d\tau_2.
\]
To prove the formula

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\text{Re}(\Lambda(x, y, y) - \Lambda(z, y, y) + \Lambda(y, x, x) - \Lambda(z, x, x) + \Lambda(z, y, x) + \Lambda(z, x, y)
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we differentiate the LHS with respect to \(x_2\).

We get a sum of double integrals of the form

\[
\int_{0}^{\infty} E_{a}^{(2)}(\tau) d\tau \int_{\tau}^{\infty} E_{b}^{(2)}(\tau_2) E_{c}^{(1)}(\tau_2) d\tau_2.
\]

Miracle: The (complicated) linear combination of products \(E_{b}^{(2)} E_{c}^{(1)}\) is actually an Eisenstein series of weight 3!

This means that we have a \textit{double Eisenstein value}. 
The double Eisenstein values can be computed using the \textit{Rogers-Zudilin method}. We get

\[
\frac{d}{dx_2} (\text{LHS}) = \text{sum of } L\text{-values } L'(G_a^{(1)}G_b^{(2)}, 0)
\]

for some (other) Eisenstein series \( G^{(1)} \) and \( G^{(2)} \).
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for some (other) Eisenstein series \( G^{(1)} \) and \( G^{(2)} \).

This can be integrated to

\[ \text{LHS} = \text{sum of } L\text{-values } L'(G^{(1)}_a G^{(1)}_b, -1) \]

We arrive at our \( L\)-value \( L'(F_{x,y}, -1) \).
The double Eisenstein values can be computed using the Rogers-Zudilin method. We get
\[ \frac{d}{dx_2} (\text{LHS}) = \text{sum of } L\text{-values } L'\left( G_a^{(1)} G_b^{(2)}, 0 \right) \]
for some (other) Eisenstein series $G^{(1)}$ and $G^{(2)}$.

This can be integrated to
\[ \text{LHS} = \text{sum of } L\text{-values } L'\left( G_a^{(1)} G_b^{(1)}, -1 \right) \]

We arrive at our $L$-value $L'\left( F_{x,y}, -1 \right)$.

**Remark**
We have no good understanding of the $\zeta(3)$ term in the formula.
The proof of the theorem also builds on:

- The Siegel modular units $g_x$ for $x \in (\mathbb{Z} / N \mathbb{Z})^2$ on the modular curve $Y(N) = \Gamma(N) \backslash \mathcal{H}$
- Milnor symbols $\{g_x, g_y\}$ in $K_2(Y(N)) \otimes \mathbb{Q}$
- Three-term relations: if $x + y + z = 0$ then
  \[ \{g_x, g_y\} + \{g_y, g_z\} + \{g_z, g_x\} = 0. \]
- We can actually find a “triangulation”
  \[ g_x \wedge g_y + g_y \wedge g_z + g_z \wedge g_x = \sum_i m_i \cdot u_i \wedge (1 - u_i) \]
  where $u_i$ and $1 - u_i$ are modular units, and $m_i \in \mathbb{Q}$.
- This triangulation leads to an element of $K_4(Y(N)) \otimes \mathbb{Q}$. 
This should extend in higher weight: for $k \geq 0$ and $x \in (\mathbb{Z}/N\mathbb{Z})^2$, there is the *Eisenstein symbol*

$$\text{Eis}^k(x) \in K_{k+1}(E(N)^k) \otimes \mathbb{Q}$$

where $E(N)^k$ is the $k$-fold fibre product of the universal elliptic curve $E(N)$ over the modular curve $Y(N)$.

**Definition**
For $k, \ell \geq 0$ and $x, y \in (\mathbb{Z}/N\mathbb{Z})^2$, define

$$X^k Y^\ell(x, y) = p_1^* \text{Eis}^k(x) \cup p_2^* \text{Eis}^\ell(y) \in K_{k+\ell+2}(E(N)^{k+\ell}) \otimes \mathbb{Q},$$

where $p_1 : E^{k+\ell} \to E^k$ and $p_2 : E^{k+\ell} \to E^\ell$ are the projections.
Conjecture

Let $k, \ell \geq 0$ and $x, y, z \in (\mathbb{Z}/N\mathbb{Z})^2$ with $x + y + z = 0$. Then

$$X^k Y^\ell(x, y) + X^\ell(-X - Y)^k(y, z) + Y^k(-X - Y)^\ell(z, x) = 0.$$
Conjecture

Let $k, \ell \geq 0$ and $x, y, z \in (\mathbb{Z}/N\mathbb{Z})^2$ with $x + y + z = 0$. Then

$$X^k Y^\ell (x, y) + X^\ell (-X - Y)^k (y, z) + Y^k (-X - Y)^\ell (z, x) = 0.$$

- One should be able to prove this in Deligne cohomology.
- Induction on the weight, using differentiation with respect to the parameters of the Eisenstein symbols.
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- Open question: what is the triangulation?
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- One should be able to prove this in Deligne cohomology.
- Induction on the weight, using differentiation with respect to the parameters of the Eisenstein symbols.
- Open question: what is the triangulation?
- In this range, Deligne cohomology is just de Rham cohomology, so this amounts to say that a particular differential form is exact. Can we make explicit a primitive?
Beyond the reach of current technology

Conjecture (Rodriguez Villegas, 2003)

\[ m(1 + x_1 + x_2 + x_3 + x_4) = -L'(f, -1) \]
\[ m(1 + x_1 + x_2 + x_3 + x_4 + x_5) = -8L'(g, -1) \]

for modular forms \( f \in S_3(\Gamma_1(15)) \) and \( g \in S_4(\Gamma_0(6)) \).
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for modular forms \( f \in S_3(\Gamma_1(15)) \) and \( g \in S_4(\Gamma_0(6)) \).

Conjecture (B.–Pengo, 2023)

\[ m(xyt + xzt + yzt + xy + xz - yz - yt + zt - y + z - t + 1) = \frac{1}{6}L'(E, -2) \]

where \( E = 32a2 \) is an elliptic curve of conductor 32.
How we found the polynomial

Take \( P(x, y, z, t) \) of the form

\[
P = a(x, y) + b(x, y)z + c(c, y)t + d(x, y)zt.
\]

Eliminating \( t \) in \( P(x, y, z, t) = P(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{t}) = 0 \) gives

\[
W_P : A(x, y)z^2 + B(x, y)z + C(x, y) = 0.
\]

Want: \( \Delta = B^2 - 4AC \) is a square \( \delta(x, y)^2 \) in \( \mathbb{Q}(x, y) \).

Then \( W_P = W_1 \cup W_2 \) with

\[
W_1 \cap W_2 : \delta(x, y) = 0.
\]

We look for \( a, b, c, d \) such that \( W_1 \cap W_2 \) is an elliptic curve.
Numerical computation of $m(P)$

**Rodriguez Villegas:** $2m(P) = \log k - \int_0^{1/k} \phi_P(u)du$ where $k$ is the constant coefficient of $P(x, y, z, t)P\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{t}\right)$ and

$$\phi_P(u) = \frac{1}{(2\pi i)^n} \int_{\Gamma^n} \frac{Q}{1-uQ} \cdot \frac{dx}{x} \frac{dy}{y} \frac{dz}{z} \frac{dt}{t}$$

with $Q = P(x, y, z, t)P\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{t}\right) - k$.

**Pengo–Ringeling:** Using creative telescoping, one can find a polynomial ODE satisfied by $\phi_P$. This takes a long time, but then $m(P)$ can be computed quickly with high precision.