Poisson brackets and flat connections

joint with F. Naef, M. Ren

Plan:

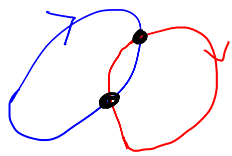
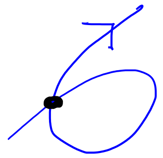
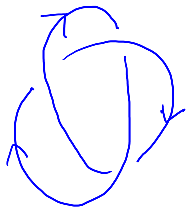
① Goldman bracket & Hitchin Theorem

② Poisson brackets of regularized holonomies

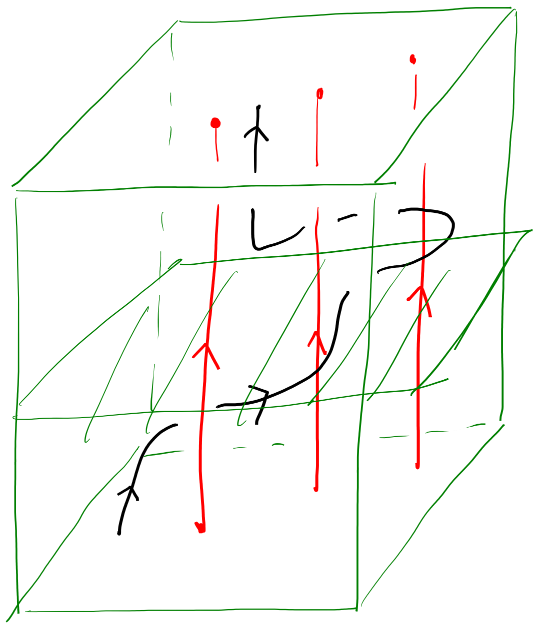
③ Ideas of proof:

- Generalized pentagon
- Turaev cobracket

Big picture

Topic	Easy	Hard
2D Topology		
3D Topology		$\Phi\Phi\Phi = \Phi\Phi$ pentagon
Physics	3D Chern-Simons $\mathrm{Tr}\left(AdA + \frac{2}{3} A^3\right)$	?
Lie theory	$\log(e^x e^y) = x + y + \frac{1}{2}[x,y] + \dots$	$\det^{1/2}\left[\frac{e^{ad_x}-1}{ad_x}\right]$
Number theory	—	$\zeta(k_1, \dots, k_m) = \sum \frac{1}{n_1^{k_1} \cdots n_m^{k_m}}$
Geometry	flat connections & Poisson brackets	BV

Today:



3D Chern-Simons with Wilson lines

- $t = \text{const}$

$\Downarrow$

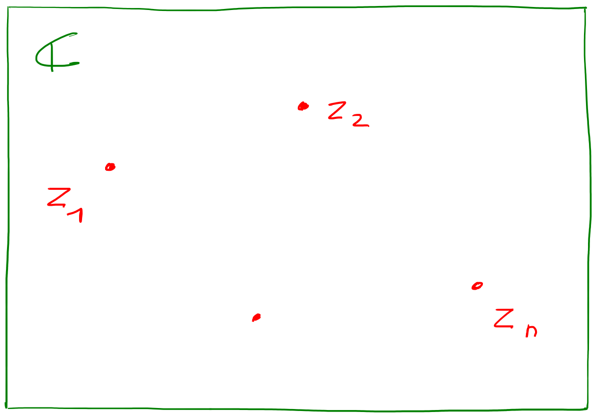
Hamiltonian picture

- $\hbar \to 0$, $k \to \infty$

$\Downarrow$

Poisson brackets

$$\Sigma = \mathbb{C} \setminus \{z_1, \dots, z_n\}$$



$$\frac{dz}{z-z_j}$$

$$\nabla = d - \sum_{j=1}^{n} X_j \, d\log(z-z_j)$$

flat connection with regular singularities

$$X_j \in \mathrm{Mat}_N(\mathbb{C})$$

KKS Poisson bracket

$$X = \sum_{ab} X_{ab} E_{ab} = \begin{pmatrix} X_{11} & & X_{1n} \\ & \cdots & \\ X_{n1} & & X_{nn} \end{pmatrix}$$

$$\{ X_{ab}, X_{cd} \} = \delta_{bc} X_{ad} - \delta_{ad} X_{cb}$$

- $\mathfrak{g} = \mathrm{Mat}_N(\mathbb{C}) \Rightarrow \mathfrak{g}^*$ Poisson

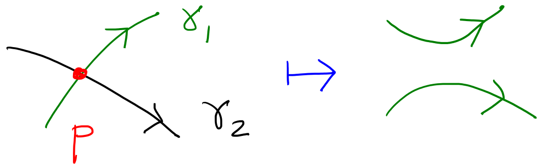
Remark: grading $\deg(X) = 2$

$$\Rightarrow \deg(\{\cdot,\cdot\}) = -2$$

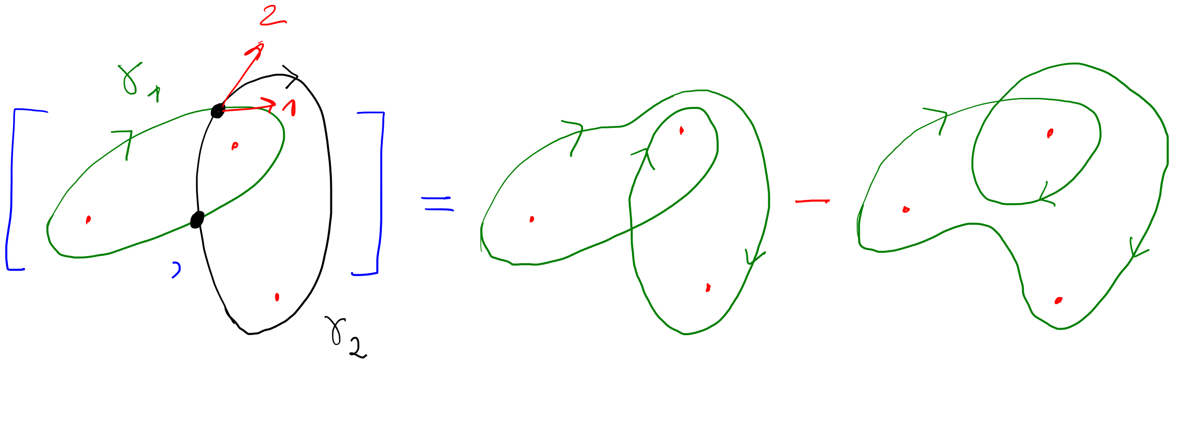
Goldman bracket

$\mathcal{G}(\Sigma) = \mathbb{C}\langle$ homotopy classes of oriented loops on $\Sigma \rangle$

$$[\gamma_1, \gamma_2] = \sum_{p \in \gamma_1 \cap \gamma_2} \overset{\pm 1}{\overset{\shortparallel}{\varepsilon_p}} \left(\gamma_1 \#_p \gamma_2\right)$$



Thm [Goldman] This is a well defined Lie bracket



Goldman functions

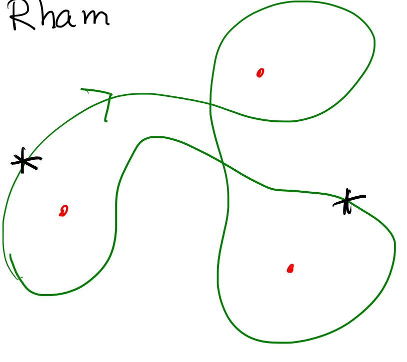
$$f_\gamma(X) = \mathrm{Tr}\ \mathrm{Hol}(\nabla, \gamma)$$

Theorem [Hitchin, ...]

$$\boxed{\{f_{\gamma_1}, f_{\gamma_2}\}_{KKS} = 2\pi i\ f_{[\gamma_1, \gamma_2]}}$$

de Rham

Betti

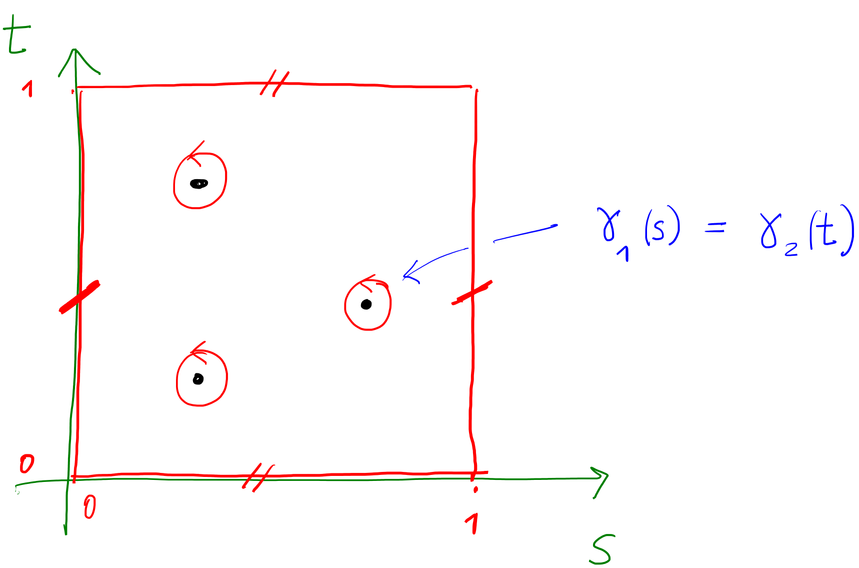


(Idea of) proof:

- $z = \gamma_1(s)$, $w = \gamma_2(t)$

- $\dfrac{d}{dp} \operatorname{Tr} \operatorname{Hol}(\nabla(p), \gamma) =$ parameter

$$= \int_\gamma \operatorname{Tr}\left(\operatorname{Hol}_s \frac{\partial \nabla(p)_s}{\partial p}\right) ds$$



Integrand $= d$ (meromorphic)

Cauchy $\Rightarrow$ Goldman

Regularized holonomy

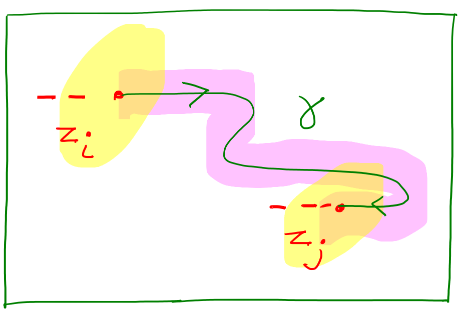
Fact : for X small enough

$\exists!\ \Psi_j(z)$ flat $\nabla \Psi_j = 0$

s.t. $\Psi_j(z) \underset{z \to z_j}{\sim} \exp(X_j \log(z - z_j))$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

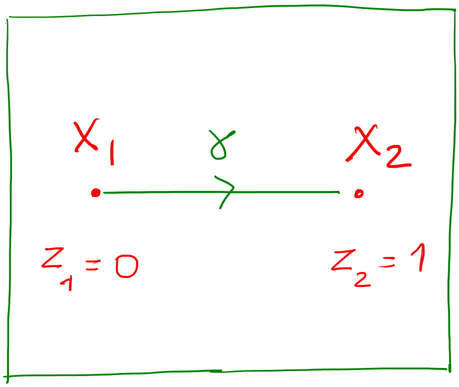
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (z-z_j)^{X_j}$



Def : $\mathrm{Hol}^{\mathrm{reg}}(\nabla, \gamma) = \Psi_j(z)^{-1} \Psi_i(z)$

- analytic continuation along γ
- independent of z

Example: $n = 2$



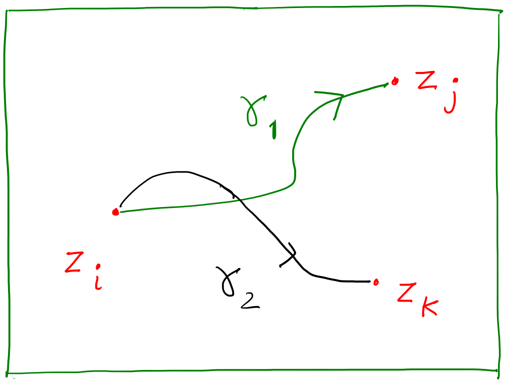
$$\Phi(x_1, x_2) = \mathrm{Hol}^{reg}(\nabla, \gamma)$$

= Knizhnik-Zamolodchikov associator [Drinfeld]

- Question:

$$\{ \mathrm{Hol}^{reg}(\gamma_1)_{ab} , \mathrm{Hol}^{reg}(\gamma_2)_{cd} \} = ?$$

Enough to consider:



Notation: $\mathrm{Hol}^{reg}(\gamma_i) = H(\gamma_i)$

Theorem [ANR]

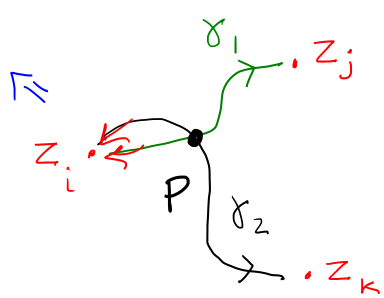
quasi-Poisson bracket

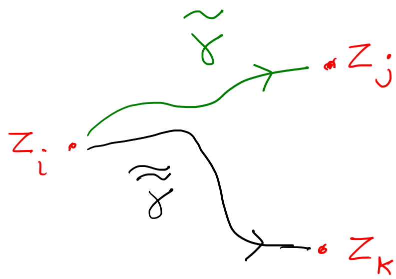
$$\{H(\gamma_1)_{ab}\,,\,H(\gamma_2)_{cd}\} =$$

$$= 2\pi i \left\{ \sum_{p\in\gamma_1\cap\gamma_2} \tilde{H}_{ad}\,\tilde{\tilde{H}}_{cb} \pm \frac{1}{2} H(\gamma_1)_{ad}\, H(\gamma_2)_{cb} \right\}$$

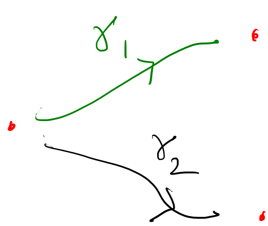
$$+ \left[\left(H(\gamma_1)\otimes H(\gamma_2)\right) \rho\left(2\pi i X_i\right) \right]_{ac,bd}$$

$\pm\frac{1}{2} \Leftarrow$





$\pm\frac{1}{2}$ depends on the order of ends at z_i



$$1 - \frac{1}{2} = +\frac{1}{2}$$

- ρ = classical dynamical r-matrix

for $X = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_N \end{pmatrix}$

- $$\rho(X) = \sum_{ab} \varphi(x_a - x_b)\, E_{ab} \otimes E_{ba}$$

- $$\varphi(s) = \frac{1}{2}\,\frac{e^s+1}{e^s-1} - \frac{1}{s}$$

$$= \sum_{k=2}^{\infty} \frac{B_k}{k!}\, s^{k-1}$$

Previous work: Korotkn - Samtleben '96

$\Leftarrow$ Massuyeau '15
Naef '16]

Ideas of proof:

① Generalized pentagon equation

$$\mathfrak{t}_n = \text{free Lie}\left\langle t_{ij}=t_{ji},\ t_{ii}=0,\ i,j=1,\dots,n\right\rangle \Big/ \begin{array}{l} \cdot\ [t_{ij}, t_{kl}] = 0 \\ \cdot\ [\underbrace{t_{ij}+t_{ik}}_{= t_{i,jk}}, t_{jk}] = 0 \end{array}$$

4T-relation

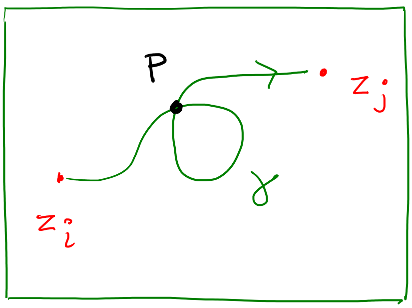
$$\mathfrak{t}_{n+1} \supset \langle \underset{\substack{\parallel \\ x_1}}{t_{1,n+1}}, \dots, \underset{\substack{\parallel \\ x_n}}{t_{n,n+1}} \rangle \quad \text{free}$$

Theorem [Drinfeld]

Let $\Phi = \mathrm{Hol}^{reg}(x_1, x_2, \underset{0}{\bullet} \longrightarrow \underset{1}{\bullet})$

$$\Rightarrow \quad \underset{\substack{\parallel \\ \Phi(t_{2,3},\, t_{3,4})}}{\Phi^{2,3,4}}\ \Phi^{1,23,4}\ \Phi^{1,2,3} = \underset{\substack{\parallel \\ \Phi(t_{12,3},\, t_{3,4})}}{\Phi^{12,3,4}}\ \Phi^{1,2,34}$$

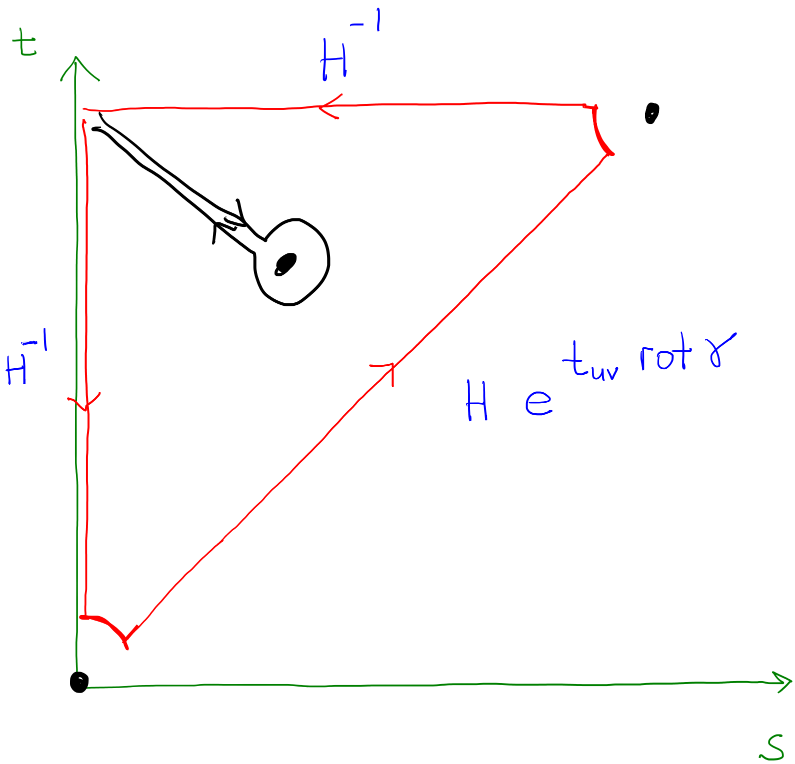
Theorem [ANR]



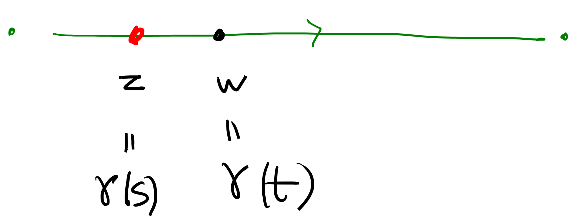
$$H = \mathrm{Hol}^{reg}(\nabla, \gamma)$$

$$\Rightarrow \left(H^{iu,v,j}\right)^{-1} \Phi^{u,v,j} H^{i,uv,j} e^{t_{uv}\,\mathrm{rot}(\gamma)} \Phi^{i,u,v} \left(H^{i,u,vj}\right)^{-1} =$$

$$= \prod_{p \in \gamma \cap \gamma} \left(C_p \, e^{2\pi i\, t_{uv}} C_p^{-1} \right)$$



$$\gamma(s) = z \quad , \quad \gamma(t) = w$$



(2) $A = \mathbb{C}\langle\langle x_1, \dots, x_n \rangle\rangle$

(completed) free associative algebra

- double bracket

$$\{\{x_i, x_j\}\} = \delta_{ij}(1 \otimes x_i - x_i \otimes 1)$$

$\Rightarrow$ KKS Poisson bracket

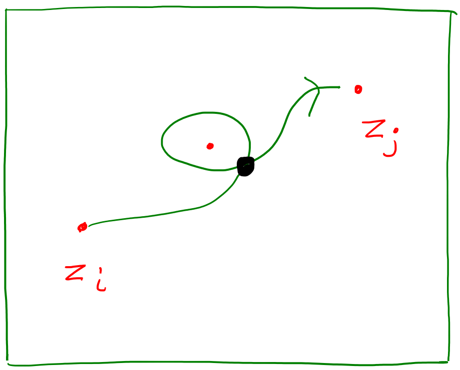
- coaction map

$$\mu : A \longrightarrow A/[A,A] \otimes A$$

$$\mu(x_i) = 0$$

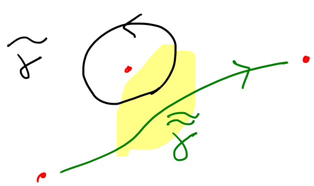
Fact : $\boxed{\mu \quad \text{determines} \quad \{\{\cdot, \cdot\}\}}$

③ Pentagon $\Rightarrow$ μ



$$\mu\left(\mathrm{Hol}^{\mathrm{reg}}(\nabla, \gamma)\right) =$$

$$= \mathrm{Tr}(\tilde{H}) \cdot \tilde{\tilde{H}} + \zeta\text{-terms}$$



$\Uparrow$

pentagon r.h.s.

$\zeta(2k+1)$

[Turaev

Kawazumi – Kuno]

$\Uparrow$

pentagon l.h.s.

Thank you !