Higher Bessel Functions: product formulas, Landau-Ginzburg type models and corresponding integer sequences

Ilia Gaiur

[1] International Group de Travail on differential equations in Paris
[2] University of Toronto

21 Apr 2023

Object of study and motivation

Joint work with Volodya Rubtsov and Duco van Straten

$$\phi_N(x)\phi_N(y) = \int_\gamma K(x, y|z)\phi_N(z)dz$$

$$\phi_N(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!^n} \quad - \quad N\text{-Bessel function}$$

General Context

- Non-Abelian Abel theorem
- Kontsevich type polynomials
- Langlands program
- Formal group laws and generalized shifts

Papers :

- M.Kontsevich *Notes on motives in finite characteristic* (2009)
- P.Etingof, E.Frenkel and D.Kazhdan *Hecke operators and analytic Langlands correspondence for curves over local fields* (2021)
- V.Golyshev, A.Mellit, V.Rubtsov and D.van Straten *Non-abelian Abel's theorems and quaternionic rotation* (2021).
- M.Kontsevich and A.Odesskii *Multiplication kernels* (2021)

Plan

- Motivation : non-Abelian Abel's theorem
- Clausen duplication kernel $K(x, x|z)$ and corresponding Picard-Fuchs operators
- Pencils of Landau-Ginzburg type models as a deformation of the Clausen formulas
- Frobenius type solution for general kernels and integer sequences
- Relation to the Buchtaber 2-valued group laws (joint work with V.M. Buchstaber)

Group law as an integral identity

- $(\mathbb{R}^\star, \cdot)$ is given by

$$\begin{array}{ccccc}
\int_1^x \frac{dt}{t} & + & \int_1^y \frac{dt}{t} & = & \int_1^{z=xy} \frac{dt}{t} \\
\log(x) & + & \log(y) & = & \log(xy) \\
x & \text{``}+\text{''} & y & = & xy
\end{array}$$

- S^1

$$\begin{array}{ccccc}
\int_1^x \frac{dt}{\sqrt{1-t^2}} & + & \int_1^y \frac{dt}{\sqrt{1-t^2}} & = & \int_1^z \frac{dt}{\sqrt{1+t^2}} \\
\arcsin(x) & + & \arcsin(y) & = & \arcsin(z) \\
x & \text{``}+\text{''} & y & = & x\sqrt{1-y^2} + y\sqrt{1-x^2}
\end{array}$$

- Elliptic integrals

$$\begin{array}{ccccc}
\int_1^x \frac{dt}{\sqrt{1-t^4}} & + & \int_1^y \frac{dt}{\sqrt{1-t^4}} & = & \int_1^z \frac{dt}{\sqrt{1-t^4}} \\
x & \text{``}+\text{''} & y & = & \frac{x\sqrt{1-y^4}+y\sqrt{1-x^4}}{1+x^2y^2}
\end{array}$$

Non-Abelian Abel theorem

Non-Abelian Abel theorem

Abelian		**Non-Abelian**
line bundles	$\rightarrow$	vector bundles
1-st order ODE		N-th order ODE

Study multiplication formulas for the equations

$$\mathcal{L}_x \Psi = \lambda \Psi$$

$$\phi_0(\lambda; x)\phi_0(\lambda; y) = \int K(x, y|z)\phi_0(\lambda; z)dz$$

where ϕ_0 is a certain analytical solution

Sonine-Gegenbauer formula

N. Sonine (1880) *Math. Ann.* XVI
L. Gegenbauer (1884) *Weiner Sitzungsberichte,* LXXXVIII

$$J_0(x)J_0(y) = \frac{1}{2\pi}\int_{x-y}^{x+y} \frac{J_0(z)dz}{\sqrt{x^4+y^4+z^4-2x^2y^2-2x^2z^2-2y^2z^2}}$$

$x < y \in \mathbb{R}$.

G. N. Watson *A Treatise on the Theory of Bessel Functions*
N. I. Vilenkin *Special Functions and the Theory of Group Representations*

Higher Bessel equations

We consider

$$\phi_N(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!^N}$$

Analytical solution of

$$[\theta^N - x]\psi = 0, \quad \theta = \frac{xd}{dx}$$

Laplace transform

$$f_N(z) = \sum_{k=0}^{\infty} \frac{(Nk)!}{k!^N} z^k$$

Duplication kernels via multiplicative convolution

Clausen duplication formula

$$\phi^{(N)}(x)\phi^{(N)}(x) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!^N}\right)^2 = \sum_{n=0}^{\infty} \frac{c_n}{n!^N} x^n, \quad c_n = \sum_{k=0}^{n} \binom{n}{k}^N$$

Hadamar product/multiplicative convolution

$$\phi^{(N)}(x)^2 = \phi^{(N)}(x) \star \sum_{n=0}^{\infty} c_n x^n =$$

$$= \phi^{(N)}(x) \star K(x) = \frac{1}{2\pi i} \oint K(x/t)\phi^{(N)}(t)dt/t$$

$K(x)$ is a generating function of c_n

Binomial coefficient sums. Franel numbers

$$K^{(N)}(x) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k}^N \right] x^n$$

- $N = 2$ - Clausen duplication for Bessel, Picard-Fuchs for 2 points scheme
$$K^{(2)}(x) = \frac{1}{\sqrt{1-4x}}$$
- $N = 3$ - Apéry like sequence of type A (D. Zagier, *Groups and symmetries* 47, 349-366). Heun function
$$x(x+1)(8x-1)K^{''} + (24x^2 + 14x - 1)K^{'} + (8x+2)K = 0$$
- Generic $N > 1$: period from Landau-Ginzburg model
$$W^{(N)} = \prod_{i=1}^{N-1}(1+X_i) + \prod_{i=1}^{N-1}(1+X_i^{-1})$$

Landau-Ginzburg type models

Kernel is a period

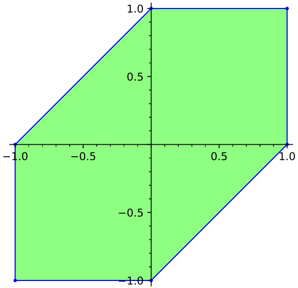
$$K^{(N)}(t) = \frac{1}{(2\pi i)^{N-1}} \oint \frac{dX_1}{X_1} \frac{dX_2}{X_2} \cdots \frac{dX_{N-1}}{X_{N-1}} \frac{1}{1 - tW^{(N)}}$$

$$W^{(N)} = \prod_{i=1}^{N-1} (1 + X_i) + \prod_{i=1}^{N-1} (1 + X_i^{-1})$$

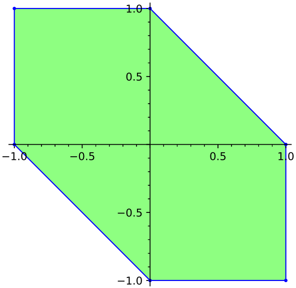
of a family in $(\mathbb{C}^{\star})^{N-1}$

$$W^{(N)}(X_1, X_2, \ldots X_{N-1})t - 1 = 0$$

N=3 Polytopes



(a) Newton polytope for $W^{(3)}$



(b) Dual polytope for $W^{(3)}$

Picard-Fuchs type operators

$$
\begin{aligned}
N=2 \quad & (2^2 t-1)\,\theta_t + 2t\\
N=3 \quad & (2^3 t-1)\,(t+1)\,\theta_t^2 + t\,(16t+7)\,\theta_t + 2t(4t+1)\\
N=4 \quad & (2^4 t-1)\,(4t+1)\,\theta_t^3 + 6t(32t+3)\theta_t^2 + 2t(94t+5)\theta_t + 2t(30t+1)\\
N=5 \quad & (2^5 t-1)\,(4t-7)^2\,(t^2-11t-1)\theta_t^4 +\\
& 2t(4t-7)(256t^3-2084t^2+4942t+143)\theta_t^3 +\\
& t(3072t^4-23024t^3+72568t^2-102261t-1638)\theta_t^2 +\\
& t(2048t^4-12896t^3+30072t^2-66094t-637)\theta_t +\\
& 2t(256t^4-1472t^3+1904t^2-7868t-49)
\end{aligned}
$$

Calabi-Yau like operators

$\exists$ Fuchsian operator $\mathcal{P}_N \in \mathbb{W}[x]$ which annihilates kernel :

$$\mathcal{P}_N K^{(N)}(x) = 0.$$

Zero is always a singular point

N	2	3	4	5	6
ord$\mathcal{P}_{\mathcal{N}}$	1	2	3	4	6
exp	0	$(0 \cdot 2)$	$(0 \cdot 3)$	$(0 \cdot 4)$	$(0 \cdot 5, 1 \cdot 1)$

7	8	9	10
8	10	12	15
$(0 \cdot 6, 1 \cdot 2)$	$(0 \cdot 7, 1 \cdot 3)$	$(0 \cdot 8, 1 \cdot 4)$	$(0 \cdot 9, 1 \cdot 5, 2 \cdot 1)$

5-Bessel. Picard-Fuchs operator

$$K^{(N)}(t) = 1 + 2\,t + 34\,t^2 + 488\,t^3 + 9826\,t^4 + 206252\,t^5 + O(t^6)$$

Corresponding Picard-Fuchs operator writes as

$$\begin{aligned}
&(2^5 t - 1)\,(4t - 7)^2\,(t^2 - 11t - 1)\theta_t^4 + \\
&2t(4t - 7)(256t^3 - 2084t^2 + 4942t + 143)\theta_t^3 + \\
&t(3072t^4 - 23024t^3 + 72568t^2 - 102261t - 1638)\theta_t^2 + \\
&t(2048t^4 - 12896t^3 + 30072t^2 - 66094t - 637)\theta_t + \\
&2t(256t^4 - 1472t^3 + 1904t^2 - 7868t - 49)
\end{aligned}$$

MUM at $t = 0$

5-Bessel kernel. Frobenius basis and mirror coordinate

Frobenius basis at zero is

$$y_0 = f_0, \quad f_0 = 1 + 2t + 34t^2 + 488t^3 + 9826t^4 + O\left(t^5\right)$$

$$y_1 = f_0 \ln(x) + f_1, \quad f_1 = 5t + \frac{175}{2}t^2 + \frac{4280}{3}t^3 + \frac{354205}{12}t^4 + O\left(t^5\right)$$

$$y_2 = f_0 \frac{\ln(x)^2}{2!} + f_1 \ln(x) + f_2, \quad f_2 = \frac{10}{7}t + \frac{290}{7}t^2 + \frac{120665}{126}t^3 + \frac{393775}{18}t^4 + O\left(t^5\right)$$

$$y_2 = f_0 \frac{\ln(x)^3}{3!} + f_1 \frac{\ln(x)^2}{2!} + f_2 \ln(x) + f_3 \quad f_3 = -\frac{20}{7}t - \frac{445}{14}t^2 - \frac{79040}{189}t^3 + O\left(t^4\right).$$

Mirror coordinate

$$q = \exp(y_1/y_0) = t + 5t^2 + 90t^3 + 1510t^4 + 31745t^5 + 697971t^6 + O(t^7),$$

Mirror map

$$t(q) = q - 5q^2 - 40q^3 + 115q^4 - 645q^5 - 12846q^6 + O(q^7).$$

5-Bessel kernel. Yukawa coupling

Yukawa potential is

$$Y(q) = \left(q\frac{d}{dq}\right)^2 \frac{y_2}{y_0} = 1 + \frac{10}{7}q + \frac{530}{7}q^2 + \frac{7975}{7}q^3 + \frac{196690}{7}q^4 + O\left(q^5\right),$$

Passing to the Lambert series of the following form

$$Y(q) = 1 + \frac{5}{7}\sum_{i=1}^{\infty} n_i i^3 \frac{q^i}{1-q^i},$$

Integer "instanton" numbers n_k

$$n_1 = 2,\ n_2 = 13,\ n_3 = 59,\ n_4 = 613,\ n_5 = 5943, n_6 = 75423$$

From Clausen to Sonine-Gegenbauer

$$\phi_N(x)\phi_N(rx) = \sum_{i=0}^{\infty}\sum_{k=0}^{i}\frac{r^k}{(k!(i-k)!)^N}x^i = \sum_{i=0}^{\infty} b_i^{(N)}(r)x^i.$$

Kernel then is a generating function for polynomials

$$c_i^{(N)}(r) = b_i^{(N)}(r)/a_i^{(N)} = \sum_{k=0}^{i}\left(\begin{array}{c} i \\ k \end{array}\right)^N r^k \in \mathbb{Z}[r]$$

$$K_N(t;r) = \sum c_i^{(N)}(r)t^i$$

Pencil of LG-type models

Theorem

The generating function $K_N(t, r)$ is a period for the deformed Landau-Ginzburg type model

$$W_N(r) = r \prod_{i=1}^{N-1} (1 + X_i) + \prod_{i=1}^{N-1} (1 + X_i^{-1}),$$

i.e. polynomials $c_i^{(N)}(r)$ are constant terms of powers $W_N(r)^i$.

N=4

Analytical and $\log$ solutions are

$$y_0(x) = 1 + (r+1)\,x + \left(r^2 + 16r + 1\right) x^2 + \left(r^3 + 81r^2 + 81r + 1\right) x^3 + \left(r^4 + 256r^3 + 1296r^2 + 256r + 1\right) x^4 + \dots$$

$$y_1(x) = y_0 \ln(x) + (2r+2)\,x + \left(3r^2 + 32r + 3\right) x^2 + \left(\frac{11}{3}r^3 + 189r^2 + 189r + \frac{11}{3}\right) x^3 + \left(\frac{25}{6}r^4 + \frac{2048}{3}r^3 + 3024r^2 + \frac{2048}{3}r + \frac{25}{6}\right) x^4 \dots$$

N=4

Mirror map

$$x = q - 2\,(r+1)\,q^2 + \left(5r^2 - 16r + 5\right)\,q^3 +$$

$$-\left(14r^3 - 42r^2 - 42r + 14\right)\,q^4 + \left(42r^4 - 108r^3 + 87r^2 - 108r + 42\right)\,q^5$$

$$-\left(132r^5 - 284r^4 - 332r^3 + 332r^2 - 284r + 132\right)\,q^6 + \mathrm{O}\left(q^7\right)$$

Multiplication formulas of Sonine-Gegenbauer type

Let $\phi_\lambda(x)$ be a solution of

$$\mathcal{L}_x\psi = \lambda\psi$$

Multiplication formula reads

$$\phi_\lambda(x)\phi_\lambda(y) = \int\limits_\gamma K(x,y|z)\phi_\lambda(z)dz = \langle K, \phi_\lambda\rangle$$

Then

$$\begin{array}{ccccc}
\mathcal{L}_x[\phi_\lambda(x)\phi_\lambda(y)] & = & \mathcal{L}_y[\phi_\lambda(x)\phi_\lambda(y)] & = & \lambda\phi_\lambda(x)\phi_\lambda(y) \\
 & & & & \parallel \\
\langle K, \mathcal{L}_z\phi_\lambda\rangle & = & \langle \mathcal{L}_z^\star K, \phi_\lambda\rangle & = & \lambda\langle K, \phi_\lambda\rangle
\end{array}$$

Multiplication formulas of Sonine-Gegenbauer type

Let $\phi_\lambda(x)$ be a solution of

$$\mathcal{L}_x \psi = \lambda \psi$$

Multiplication formula reads

$$\phi_\lambda(x)\phi_\lambda(y) = \int_\gamma K(x,y|z)\phi_\lambda(z)dz = \langle K, \phi_\lambda \rangle$$

Then

$$\begin{array}{ccccc} \mathcal{L}_x[\phi_\lambda(x)\phi_\lambda(y)] & = & \mathcal{L}_y[\phi_\lambda(x)\phi_\lambda(y)] & = & \lambda\phi_\lambda(x)\phi_\lambda(y) \\ & & & & \shortparallel \\ \langle K, \mathcal{L}_z\phi_\lambda \rangle & = & \langle \mathcal{L}_z^\star K, \phi_\lambda \rangle & = & \lambda\langle K, \phi_\lambda \rangle \end{array}$$

$$\boxed{\mathcal{L}_x[K] = \mathcal{L}_y[K] = \mathcal{L}_z^\vee[K]}$$

N-Bessel kernels

$$\left[\frac{d}{dx}\theta_x^{N-1} - \frac{d}{dz}\theta_z^{N-1}\right] K = 0, \quad \left[\frac{d}{dy}\theta_y^{N-1} - \frac{d}{dz}\theta_z^{N-1}\right] K = 0,$$

with initial conditions

$$K_N(0, y, z) = \frac{1}{z - y}$$

Applying Frobenius method we get

$$K(x, y, z) = \sum_{j=0}^{\infty} a_j(y, z) x^j, \quad a_0(y, z) = \frac{1}{z - y}$$

$$a_j(y, z) = \frac{1}{j^N} \frac{d}{dz} \theta_z^{N-1} [a_{j-1}(y, z)] = \frac{1}{j^N} \frac{d}{dz} (\theta_z^{N-1})^j \frac{1}{z - y}$$

N=2

Kernel reads

$$K_2(x,y;z) \simeq \frac{1}{z-y} + \frac{(y+z)}{(z-y)^3}x + \frac{(y^2+4yz+z^2)}{(z-y)^5}x^2 + $$
$$+$$
$$\frac{(y^3+9y^2z+9yz^2+z^3)}{(z-y)^7}x^3 + \frac{(y^4+16y^3z+36y^2z^2+16yz^3+z^4)}{(z-y)^9}x^4 + \dots$$

Operator which annihilates series

$$\left(x^2+y^2+z^2-2xy-2xz-2yz\right)\frac{d}{dx}K + (x-z-y)\,K = 0$$

Gives Sonine-Gegenbauer type kernel

N=2

Consider again kernel for 2-Bessel

$$K(x,y;z) \simeq \frac{1}{z-y} + \frac{(y+z)}{(z-y)^3}x + \frac{(y^2+4yz+z^2)}{(z-y)^5}x^2 +$$
$$+\frac{(y^3+9y^2z+9yz^2+z^3)}{(z-y)^7}x^3 + \frac{(y^4+16y^3z+36y^2z^2+16yz^3+z^4)}{(z-y)^9}x^4 + \dots$$

Rewrite $K(x,y|z)$ as

$$K(x,y|z) = \sum_{i=0}^{\infty} P_i(y,z)\frac{x^i}{(z-y)^{2i+1}}, \quad P_i(y,z) \in \mathbb{Z}[y,z]$$

Statement : $P_i(x,y)$ are monic palindromic

$$P_i(y,z) = \sum_{k+n=i} T^{(i)}_{k,n} y^k z^n, \quad \mathbf{T^{(i)}_{k,n}} = \mathbf{T^{(i)}_{n,k}} \quad \mathbf{T^{(i)}_{i,0}} = \mathbf{T^{(i)}_{0,i}} = \mathbf{1}$$

Generating function for the "square" of Pascal triangle

N=2

$$
\begin{array}{ccccc}
 & & T^{(0)}_{0,0} & & \\
 & T^{(1)}_{1,0} & & T^{(1)}_{1,0} & \\
T^{(2)}_{2,0} & & T^{(2)}_{1,1} & & T^{(2)}_{0,2} \\
 & & \cdots & &
\end{array}
=
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 4 = 2^2 & & 1 & & & & \\
 & & & 1 & & 9 = 3^2 & & 9 & & 1 & & & \\
 & & 1 & & 16 & & 36 & & 16 = 4^2 & & 1 & & \\
 & 1 & & 25 & & 100 & & 100 & & 25 & & 1 & \\
1 & & 36 & & 225 & & 400 & & 225 & & 36 & & 1 \\
 & & \cdots & & & & \cdots & & & & \cdots & &
\end{array}
$$

$$
T^{(m)}_{k,m-k} = \frac{1}{k!}\frac{d^k}{dt^k}\left[(1-t)^{2m+1}\sum_{j=0}^{k}\binom{m+j}{j}^2 t^j\right] :=
$$

$$
= k\text{-th coefficient of } (1-t)^{2m+1}\sum_{j=0}^{k}\binom{m+j}{j}^2 t^j
$$

N-Bessel numerology

Kernel is

$$K^{(N)}(x,y|z) = \sum_{m=0}^{\infty} P_m^{(N)}(y,z)\frac{x^m}{(z-y)^{mN+1}}, \quad \deg P_m^{(N)} = m(N-1).$$

Statement : $P_i(x,y)$ are monic palindromic

$$P_i^{(N)}(y,z) = \sum_{k+n=i(N-1)} {}^{(N)}T_{k,n}^{(i)} y^k z^n, \quad \mathbf{T_{k,n}^{(i)}} = \mathbf{T_{n,k}^{(i)}} \quad \mathbf{T_{i,0}^{(i)}} = \mathbf{T_{0,i}^{(i)}} = \mathbf{1}$$

Coefficients are

$${}^{(N)}T_{k,m(N-1)-k}^{(m)} = \frac{1}{k!}\frac{d^k}{dt^k}\left[(1-t)^{Nm+1}\sum_{j=0}^{k}\binom{m+j}{j}^N t^j\right] :=$$

$$= k\text{-th coefficient of } (1-t)^{Nm+1}\sum_{j=0}^{k}\binom{m+j}{j}^N t^j$$

SOME OF THESE NUMBERS ARE KNOWN

A181544 for $N=3$

A262014 for $N=4$

N=3

For $N = 3$ kernel reads as

$$K(x, y, z) = -\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; -\frac{27xyz}{(x+y-z)^3}\right)}{x + y - z}.$$

with non-apparent singularity loci

$$x(2x + z - y)((x - z + y)^3 + 27xyz) = 0$$

Expanding into series

$$\frac{1}{x+y} + \frac{x^2 - 4xy + y^2}{(x+y)^4} z + \frac{x^4 - 20x^3y + 48x^2y^2 - 20x\,y^3 + y^4}{(x+y)^7} z^2 + \\ + \frac{x^6 - 54x^5y + 405x^4y^2 - 760x^3y^3 + 405x^2y^4 - 54x\,y^5 + y^6}{(x+y)^{10}} z^3 + \mathrm{O}\left(z^4\right)$$

Singularity loci for N-Bessel kernels

$$N=2 \quad x^2+y^2+z^2-2xy-2yz-2xz$$

$$N=3 \quad (x-z+y)^3+27xyz$$

$$N=4 \quad x^4-4x^3y-4x^3z+6x^2y^2-124x^2yz+6x^2z^2-4y^3x$$
$$-124xy^2z-124xyz^2-4z^3x+y^4-4y^3z+6y^2z^2-4yz^3+z^4$$

Substitution $x=u^N, y=v^N, z=w^N$

$$N=2 \quad (u+v+w)(u-v+w)(u+v-w)(u-v-w)$$

$$N=3 \quad (u+v-w)(u^2-uv+uw+v^2-2vw+w^2)$$
$$(u^2-uv-2uw+v^2+vw+w^2)(u^2-uv+uw+v^2+vw+w^2)$$
$$(u^2+2uv+uw+v^2+vw+w^2)$$

$$N=4 \quad (u+v+w)\,(u+v-w)\,(u-v+w)\,(u-v-w)$$
$$\left(u^2+v^2-2vw+w^2\right)\left(u^2+v^2+2vw+w^2\right)$$
$$\left(u^2+2uw+v^2+w^2\right)\left(u^2-2uw+v^2+w^2\right)$$
$$\left(u^2+2uv+v^2+w^2\right)\left(u^2-2uv+v^2+w^2\right)$$

Conjecture

The singular locus of the $D-$ module of H_N consists of the coordinate triangle $xyz = 0$, together with a rational curve R_N of degree N with $\frac{(N-1)(N-2)}{2}$ double points, which intersects the coordinate triangle only in the three mid-points $(0:1:1), (1:0:1), (1:1:0)$, defined by a symmetric polynomial

$$\Delta_N(x, y, z) = x^N + y^N + z^N + \ldots,$$

determined by the property

$$\Delta_N(u^N, v^N, w^N) = \prod_{\omega,\eta}(x + \omega y + \eta z),$$

where in the product ω and η run over the $N-$th roots of unity.

Two-valued formal group laws

Let R be a commutative ring with unit. ($R = \mathbb{Z}, \mathbb{R}, \mathbb{C}$, or the polynomial rings over $\mathbb{Z}, \mathbb{R}, \mathbb{C}$. The equation $w = \Phi(u, v)$ where $\Phi(u, v)$ is a formal series, defines a classical formal group law over the ring R. This means that :

- $\Phi(u, 0) = u$;
- $\Phi(u, \Phi(v, w)) = \Phi(\Phi(u, v), w)$.

Buchstaber family of polynomials-1

Consider the two-valued formal group with multiplication defined by the relation

$$B(x,y,z) := z^2 - \Theta_1(x,y)z + \Theta_2(x,y) = 0,$$

where

$$\Theta_1(x,y) = Z_+ + Z_-; \quad \Theta_2(x,y) = Z_+ Z_-,$$

with

$$Z_+ = \Phi(u,v)\Phi(\bar{u},\bar{v}) = |\Phi(u,v)|^2; \quad Z_- = \Phi(\bar{u},v)\Phi(u,\bar{v}) = |\Phi(\bar{u},v|^2.$$

Such two-valued group is called the **"square modulus"** of the original formal group.

Buchstaber family of polynomials-2

For the elementary formal group structure with $\Phi(u, v) = u + v$ connected with cohomology theory the two-valued group determined by

$$B(x, y, z) := x^2 + y^2 + z^2 - 2xy - 2xz - 2yz = 0.$$

This was the very first example of a two-valued group of Buchstaber–Novikov (1971).
In K-theory $\Phi(u, v) = u + v - quv$ and

$$B(x, y, z) := x^2 + y^2 + z^2 - 2xy - 2xz - 2yz - q^2 xyz = 0-$$

Cayley nodal cubic surface

Buchstaber family of polynomials-3

Buchstaber (1990) had classified the two-valued algebraic groups coming from the square modulus construction for formal groups with the multiplication law suggested by the addition theorem for Baker–Akhiezer elliptic functions by a zero locus of the following discriminant family depending in 3 parameters (a_1, a_2, a_3) :

$$B_{a_1,a_2,a_3}(x, y, z) := (x+y+z-a_2xyz)^2-4(1+a_3xyz)(xy+yz+zx+a_1xyz).$$

The parameters are expressed via the standard Weierstrass elliptic parameters g_2, g_3 and a point α on the corresponding elliptic curve $\mathcal{E}$ $v^2 = 4u^3 - g_2u - g_3$ by

$$a_1 = 3\wp(\alpha), a_2 = 3\wp(\alpha)^2 - g_2/4, a_3 = 1/4(4\wp(\alpha)^3 - g_2\wp(\alpha) - g_3).$$

Buchstaber - Veselov theorem

Theorem (Buchstaber - Veselov , 2019)
The discriminant locus multiplication law $B_{a_1,a_2,a_3}(x,y,z) = 0$ with the "elliptic parametrization" can be reduced to the addition law $X \pm Y \pm Z = 0$ via the change of variables :

$$x = \frac{1}{\wp(X) + \wp(\alpha)}, y = \frac{1}{\wp(Y) + \wp(\alpha)}, z = \frac{1}{\wp(Z) + \wp(\alpha)}.$$

The proof is based on the addition formula of Burnside determinant (1873) (Weierstrasse $\wp-$ function addition) :

$$P_{(0,\frac{-g_2}{4},\frac{-g_3}{4})}(x,y,z) := (xy + yz + zx + \frac{g_2}{4})^2 - 4(1 - \frac{g_3}{4}xyz)(x + y + z)$$

from Sonine kernel to Kontsevich polynomials

The polynomial $\Delta_2(x, y, z)$ is a very special case to a more general class of (generalized) Kontsevich polynomials :

$$\begin{aligned} P_{a,b,c}(x, y, z) &= (xy + yz + xz - b)^2 - 4(xyz + c)(x + y + z + a), \\ &= (x - y)^2 z^2 - 2((xy + b)(x + y) + 2axy + c)z \\ &\qquad + (xy - b)^2 - 4c(x + y + a, \end{aligned}$$

which is a discriminant of the quadratic trinomial :

$$t^3 + at^2 + bt + c - (t - x)(t - y)(t - z).$$