E-FUNCTIONS AND GEOMETRY

(joint work with P. Jossen)

exponential periods

$$\int_\sigma e^{-f}\, w \qquad f : X \longrightarrow \mathbb{A}^1$$

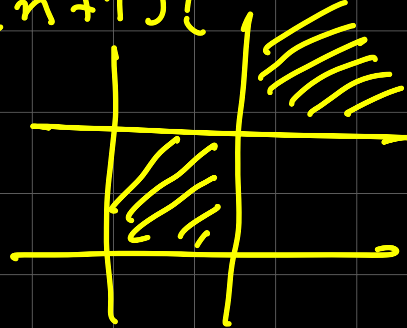
algebraic variety $/\overline{\mathbb{Q}}$

$$* \int_0^\infty e^{-x^n}\, dx = \frac{1}{n}\Gamma\left(\frac{1}{n}\right).$$

$$* \; \gamma = \iint_{\square} e^{-xy}\, dx\, dy - \int_{\Xi} e^{-xy} \frac{dx}{dy}$$

Euler's constant

$$\overbrace{\sum_{n=0}^{\infty} \frac{(-1)^n z^n}{(n+1)(n+1)!}}^{\text{value at } z=1}$$



$$* \quad \frac{\oint e^{-z(x-\frac{1}{x})} \frac{dx}{x}}{\oint \frac{dx}{x}} \qquad = \underline{\underline{2\pi i}} \cdot \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}}_{\substack{\text{value at} \\ z=1 \text{ of a} \\ \text{Bessel} \\ \text{function}}}$$

$\underline{Q}$: What can be expected in general?

E-functions

An E-function is

$$E(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \in \overline{\mathbb{Q}}[[z]]$$

such that:

(1) $\exists L \in \overline{\mathbb{Q}}[z, \frac{d}{dz}] \setminus \{0\}$

$$L \cdot E = 0 \quad \text{(holonomic)}$$

(2) $|\sigma(a_n)| \leq C^n$

for all $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, $n \geq 1$

$\mathrm{den}(a_0, \ldots, a_n) < C^n$

Variants: $\boxed{G(z) = \sum a_n z^n}$

G-function

$$\Im(z) = \sum \underline{n!\, a_n} z^n$$

$\Im$-function

Examples: (0) polynomials

(1) $a_n = 1$ $E(z) = e^z$

(2) Bessel functions

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n}$$

$$= \sum \frac{(-1)^n}{4^n} \binom{2n}{n} \frac{z^{2n}}{(2n)!}$$

(Gevrey series $\sum \frac{a_n}{n!^s} z^n$)

(1) & (2) particular examples

hypergeometric E-functions

$$F\left(\begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix} \;\middle|\; z^{q-p}\right)$$

$q > p \geq 0$

① PERIODS

$$\int_\sigma \omega$$

$\sigma : [0,1]^n \to X(\mathbb{C})$

$\omega \in H^n_{dR}(X, Y) \quad X/\bar{\mathbb{Q}}$

$\sigma \in H^B_n(X, Y) \quad Y \subset X$

Conjecture (Bombieri-Dwork conjecture)

$$\mathrm{Periods}\left[\frac{1}{\pi}\right] = \left\{ \begin{array}{l} \text{special values} \\ \text{of } G\text{-functions} \end{array} \right\}$$

$\alpha \in \overline{\mathbb{Q}}$

$G(\alpha)$ $\qquad G$

π is a period

$\frac{1}{\pi}$ not expected to be a period

$$= \underbrace{\frac{1}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{n!^3}}(-1)^n(4n+1)$$

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1)$$

$$(1)_n = n!$$

$\boxed{\subseteq}$ known

Argument

Every period can be written as

$$\int_{[0,1]^n} \underbrace{f(z_1, \dots, z_n)}_{} \, dz_1 \dots dz_n$$

power series with radius of convergence > 1

algebraic over $\overline{\mathbb{Q}}(z_1, \dots, z_n)$

$$t \mapsto \int_{[0,t]^n} f(z_1, \dots, z_n) \, dz_1 \dots dz_n$$

a G-function. (value at $t = 1$)

Ex: algebraic function $f(z)$ regular at $z=0$

$\xrightarrow[\text{Eisenstein}]{}$ f is a G-function

$\int_0^t f(z)\,dz$ also a G-function

$\int_\sigma \omega$ $\{\text{periods}\}\left[\frac{1}{\pi}\right] \simeq \mathcal{G}$

$\cap$

$\int_\sigma e^{-f}\omega$ $\left\{\begin{matrix}\text{exponential}\\ \text{periods}\end{matrix}\right\} \subset \boxed{\underline{??}}$

$\mathbb{E}$ = special values of E-functions

$$\boxed{\mathbb{E} \cap \mathcal{G} = \overline{\mathbb{Q}}}$$ Conjecture

$\Rightarrow$ $e + \pi \notin \overline{\mathbb{Q}}$

$e \cdot \pi \notin \overline{\mathbb{Q}}$

Exponential period functions

$$X \xrightarrow{f} \mathbb{A}^1$$ regular function

smooth affine algebraic over $\overline{\mathbb{Q}}$ $\dim X = n$

- $$H^n_{dR}(X, f) = \frac{\Omega^n(X)}{\operatorname{Im}(d - df \wedge \cdot)}$$

twisted dR cohomology

$dy - df \wedge y$

$\int_\sigma e^{-f} w$ Stokes's formula

$$d(e^{-f} w) = e^{-f}(\underbrace{dw - df \wedge w})$$

- Rapid decay homology

non-compact cycles on $X(\mathbb{C})$

$$H_n^{rd}(X, f) \ni \sigma = (\sigma_t)$$

$$= \lim_{t \to \infty} H_n(X(\mathbb{C}), \{\underline{\mathrm{Re}\, f \geq t}\}; \mathbb{Q})$$

(Hien-Roucairol)

$$H^n_{dR}(X, f) \otimes H_n^{rd}(X, f) \longrightarrow \mathbb{C}$$

$$([w], [\sigma]) \longmapsto \lim_{t \to \infty} \int_{\sigma_t} e^{-f} w$$

$\shortparallel$

$$\|$$

$$\int_\sigma e^{-f} \underline{\omega}$$

Make a function:

$$z \longmapsto \int_\sigma e^{-zf} \omega$$

$\mathrm{Re}\, z > 0$

<u>Theorem</u>: $F(z) = \int_\sigma e^{-zf} \omega$

extends to a linear combination of functions

$$\boxed{z^a (\log z)^b} E(z)$$

$a \in \mathbb{Q}$
$b \in \mathbb{Z}_{\geq 0}$

E-function

quasi-unipotent monodromy and $z=0$

with coefficients in

$$\overline{\mathbb{Q}}\left[\text{periods}, \underbrace{\Gamma(\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}), \gamma}\right].$$

<u>Conjecture</u>: All E-functions appear this way.

[Fourier transform of the Bombieri-Dwork conjecture]

e.g.

$${}_1F_1\left(\begin{matrix} a \\ b \end{matrix}\,\middle|\, z\right) \qquad a, b \in \mathbb{Q}.$$

$$\|$$

$$\rightarrow \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$$

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 \underbrace{e^{zu} u^{a-1} (1-u)^{b-a-1} du}_{\text{period function}}$$

very special kind of generating

$$\int_0^1 e^{zu} \underbrace{u^{a-1} (1-u)^{b-a-1}}\, du$$

$a = \frac{2}{d}$ $b = \frac{5}{d}$ $X : x^d + y^d = 1$

$$f = x^d \qquad u^{1/d} \longleftrightarrow x$$
$$(1-u)^{1/d} \longleftrightarrow y$$

Siegel asked if all E-functions are polynomial expressions in hypergeometric E-functions.

Theorem: No!

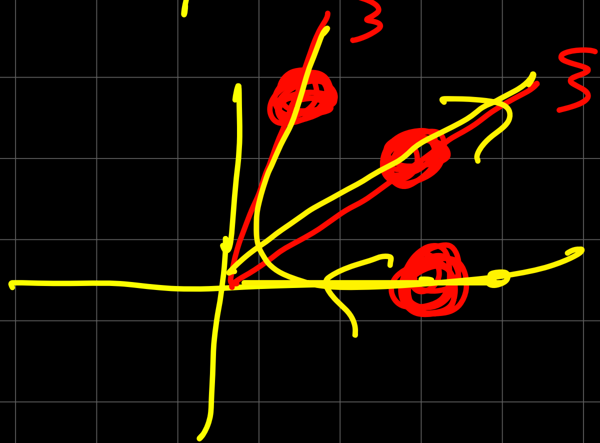
About Γ and γ:

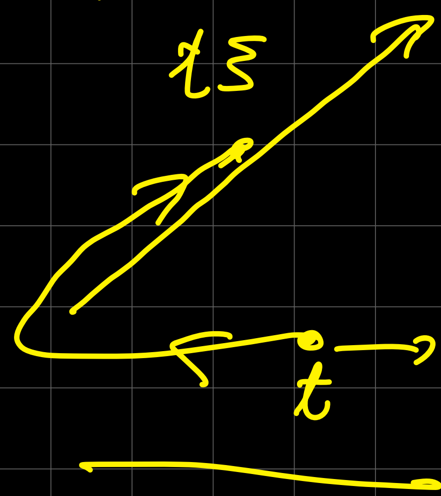
$$X = \mathbb{A}^1 \quad f = x^n$$

$$H^1_{dR}(X, f) = \mathrm{coker}\begin{pmatrix} \mathbb{Q}[x] \to \mathbb{Q}[x]\,dx \\ p \mapsto \left(p' - n x^{n-1} p\right) dx \end{pmatrix}$$

$$= \langle dx, x\,dx, \ldots x^{n-2}\,dx \rangle$$

$$H_1^{2d}(X, f) = \langle \gamma_1, \dots, \gamma_{n-1} \rangle$$





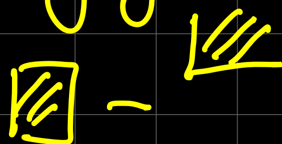
$X(\mathbb{C})$

$\xi = e^{\frac{2\pi i}{n}}$

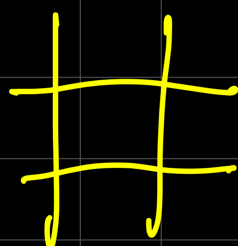
$$\int_{\gamma_i} e^{-z x^n} \underbrace{x^{j-1}\, dx}_{} = \frac{\xi^{ij} - 1}{n} \Gamma\left(\frac{j}{n}\right)$$

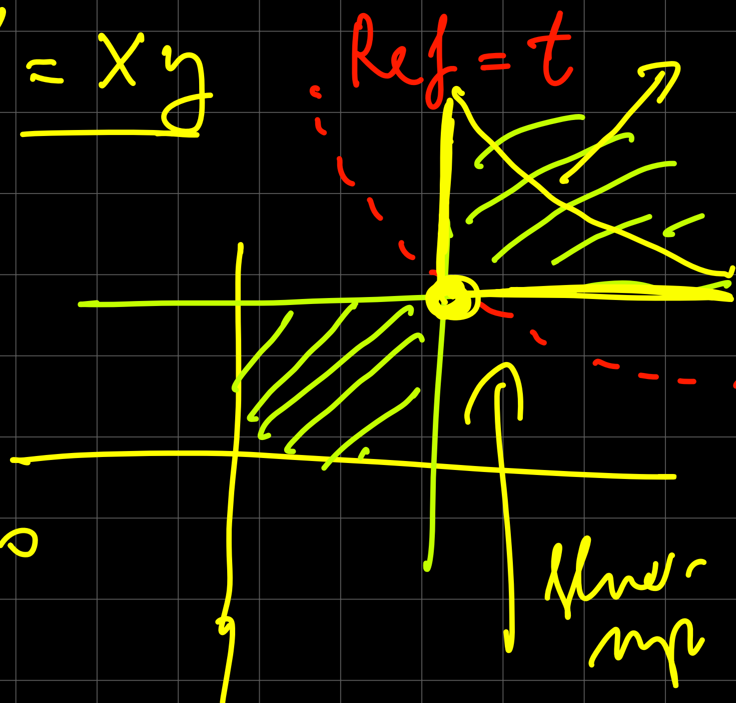
futur: $*$ $\underline{\underline{z^{j/n}}}$

$\gamma \qquad \iint e^{-2xy}\, dx\, dy$



$X = \mathbb{A}^2, \quad f = \underline{xy}$

$Y =$ 



$xy(x-1)(y-1) = 0$

$H^2(X, \underline{Y}, f)$

$\begin{pmatrix} 1 & \zeta(n) \\ 0 & (2\pi i)^n \end{pmatrix}$

dim 3

$$\begin{pmatrix} 1 & \gamma \\ 0 & 2\pi i \end{pmatrix} \quad \underline{n=1}$$

function $\ast$ $\log z$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$ Jordan block.

<u>Sketch</u>:

- consider the differential equation of $\displaystyle\int_0 e^{-zf}\,\omega$
- quasi-unipotent monodromy at $z=0$ (Y. André)

- fabricate a $\otimes$-structure in $H^1(\mathbb{A}^1, x^n)$ and $H^2(X, Y, x_y)$ with $X = \mathbb{A}^2$ the same monodromy $\square$

<u>Next?</u>

What to do with E-functions?

$$E(z) = \sum (-1)^n \, n! \, z^n$$

$$\rightsquigarrow$$

$$G(z) = \sum (-1)^n z^n = \frac{1}{z+1}$$

$$z \mapsto \int_0^\infty e^{-x} G(xz)\, dx$$

value at $z = 1$ $\quad \displaystyle\int_0^\infty \frac{e^{-x}}{x+1}\, dx$

Gompertz's constant

What can be said in general about these numbers?

The series $\displaystyle\sum_{n=0}^\infty (-1)^n n!$

converges in $\mathbb{Q}_p$ for all p.

<u>Conjecture</u>: $\sum n! \in \mathbb{Q}_p$

does not lie in $\mathbb{Q}$

No single value of ρ for which this is known.

Siegel-Shidlovsky theorem

$$\frac{d}{dz}\begin{pmatrix} E_1 \\ \vdots \\ E_n \end{pmatrix} = \underset{M_n(\overline{\mathbb{Q}}(z))}{A(z)} \begin{pmatrix} E_1 \\ \vdots \\ E_n \end{pmatrix}$$

$\alpha \in \overline{\mathbb{Q}}$ $\quad \alpha \neq 0$ $\quad$ not a pole of $A(z)$

$$\underset{\overline{\mathbb{Q}}[X_1, \dots X_n]}{P}(E_1(\alpha), \dots, E_n(\alpha)) = 0$$

$\exists\, Q \in \mathbb{Q}[z, X_1, \dots X_n]$

$Q(\alpha, \dots) = P$

$Q(z, E_1(z), \dots E_n(z)) = 0.$

$\gamma = \zeta(1)^{\text{reg}}$

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s)$$