Two applications of Grothendieck's Algebraicity Conjecture.

M. Kontsevich, IHES

Algebraicity Conjecture (A.Grothendieck)

Let (E, ∇) be an algebraic vector bundle over a smooth algebraic variety X defined over a number field K. Then (E, ∇) can be trivialized on a finite cover of X (i.e. "solvable in algebraic functions") iff the p-curvature of ∇ vanishes for all sufficiently large primes $p \gg 1$ for some model $\mathcal{X}$ of X over $\mathcal{O}_{K,S}$ where $S \subset \operatorname{Spec} K$ is a finite set of non-archimedean places.

N.Katz proved that if p-curvatures vanish for large p then the connection has only *regular* singularities, and the monodromy around any divisor at infinity is finite.

Also, in 1972 he proved Algebraicity Conjecture for connections of geometric origin (i.e. Gauss-Manin connections). The key idea is that the associated graded operator for the p-curvature with respect to Hodge filtration is the Frobenius twist of the Kodaira-Spencer operator. The vanishing of p-curvatures implies vanishing of Kodaira-Spencer operators, which implies that the positive Hermitian form coming from polarization is covariantly constant. The finiteness of the monodromy follows from the fact that the monodromy is both integral and unitary.

Weakened form of the Algebraicity Conjecture

Suppose we are given a holonomic system of linear differential equations on a smooth algebraic variety X defined over a number field K, and a regular point $x \in X(K)$ together with a system of local coordinates at x given by algebraic functions. Let (s_i) be a basis of the space of formal solutions at x. Then all s_i are algebraic iff the coefficients of *all* series (s_i) belong to $\mathcal{O}_{K,S}$ for some finite set S.

The condition of integrality of coefficients does not depend on the choice of point x, local coordinates, and the basis (s_i).

It is easy to show implications:

$$\text{Algebraicity} \implies \text{Integrality of solutions} \implies \text{vanishing of } p-\text{curvatures for } p \gg 1$$

hence the original Algebraicity Conjecture means that all 3 properties are equivalent.

Notice that there are numerous situations when only *one* of solutions has integer coefficients and the monodromy is not finite, e.g.

$$\sum_{n\geqslant 0} \frac{(dn)!}{(n!)^d} t^n \in \mathbb{Z}[[t]], \quad \text{for a given } d \geqslant 3$$

Non-linear version of Algebraicity Conjecture (V.Lunts)

Let X be a smooth algebraic variety defined over a number field K, endowed with an *integrable* distribution $\mathcal{F} \subset T_X$. The all leaves of $\mathbb{F}$ are algebraic iff for some point $x \in X(K)$ one can trivialize $\mathbb{F}$ by a formal change of variables with coefficients in $\mathcal{O}_{K,S}$ for some finite S.

Again, as in the linear case, the condition does not depend on the choice of point x.

The non-linear version is largely open, besides the homogeneous cases considered by J.-B.Bost.

The main illustration of Grothendieck's conjecture: ***hypergeometric series*** (already observed by N.Katz):

let $(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m) \in \mathbb{Q}$ be a collection of $2m$ rational numbers with pairwise distinct fractional parts $(\lfloor \alpha_i \rfloor, \lfloor \beta_i \rfloor) \in \mathbb{Q} \cap [0,1) \simeq \mathbb{Q}/\mathbb{Z}$.

Consider the hypergeometric differential equation on $\mathbb{A}^1_{\mathbb{Q}} - \{0,1\}$:

$$\Big[\prod_{i=1}^{m} (t\partial_t - \alpha_i) - \prod_{i=1}^{m} (t\partial_t - \beta_i) \cdot t \Big] f = 0$$

It has fundamental system of solutions at $t = 0$ given by

$$f_j = t^{\alpha_j} \sum_{n \geq 0} \frac{\prod_i \Gamma(1 + n + \alpha_j - \beta_i)}{\prod_i \Gamma(1 + n + \alpha_j - \alpha_i)} t^n, \qquad j = 1, \ldots, m$$

Dividing f_j by the constant $\frac{\prod_i \Gamma(1+\alpha_j - \beta_i)}{\prod_i \Gamma(1+\alpha_j - \alpha_i)}$ we obtain a series with *rational* coefficients.

Theorem (N.Katz for $m = 2$, F.Beukers-G.Heckman for general m)

Hypergeometric series is algebraic iff for any N coprime to denominators of all $(\alpha_i), (\beta_i)$ the fractional parts $\lfloor N\alpha_i \rfloor \subset \mathbb{Q}/\mathbb{Z}$ are interlaced with the fractional parts $\lfloor N\beta_i \rfloor \subset \mathbb{Q}/\mathbb{Z}$.

The interlacing criterion comes from the analysis of powers of primes in coefficients of hypergeometric series. Enough to check for finitely many values of N.

One can classify all algebraic hypergeometric series in one variable, the answer is given in terms of finite complex reflection groups.

Example (special case, ratios of products of factorials):

$$f(t) = \sum_{n \geqslant 0} \frac{(6n)!n!}{(3n)!(2n)!(2n)!} \left(\frac{t}{108}\right)^n \in \mathbb{Z}[[\frac{t}{108}]], \quad 108 = \frac{6^6 1^1}{3^3 2^2 2^2}$$

is algebraic:

$$f(t) = \frac{1}{\sqrt{1-t}} \left(\frac{(1+\sqrt{1-t})^{2/3}}{t^{5/6}} + \frac{(1+\sqrt{1-t})^{-2/3}}{t^{1/6}} \right)$$

One can prove the algebraicity criterion for hypergeometric series in a purely elementary way, *without* using Katz' result.

Lemma (from linear algebra)

The set of conjugacy classes of pairs $(T_0, T_\infty) \in U(n)^2$ of unitary operators such that $\mathrm{rk}(T_0 - T_\infty) \leqslant 1$, is in one-to-one correspondence (via $(T_0, T_\infty) \mapsto (\mathrm{Spec}\, T_0, \mathrm{Spec}\, T_\infty)$) with the set of two n-tuples of points in $U(1)$ which are (non-strictly) interlaced:

$$(e^{2\pi i \alpha_j})_{1 \leqslant j \leqslant m}, \quad (e^{2\pi i \beta_j})_{1 \leqslant j \leqslant m}$$

$$\text{where} \quad 0 \leqslant \alpha_1 \leqslant \beta_1 \leqslant \alpha_2 \leqslant \beta_2 \leqslant \ldots \leqslant \alpha_n \leqslant \beta_n < 1 \quad [\text{ or } \alpha_i \leftrightarrow \beta_i]$$

In a sense, Katz' theorem is quite easy (although the paper is somewhat long).
Up to now, the only really hard arithmetic result confirming Grothendieck conjecture is

Theorem (D.Chudnovsky and G.Chudnovsky, 1984)

Grothendieck's algebraicity conjecture holds for connections on vector bundles of rank 1.

A weakened form: if for an algebraic series $f(t) = \sum_{n\geqslant 1} a_n t^n \in \mathbb{Z}[[t]] \cap \overline{\mathbb{Q}(t)}$, the series

$$g(t) = \exp\Big(\sum_{n\geqslant 1} \frac{1}{n} a_n t^n\Big) \in \mathbb{Q}[[t]]$$

has *integer coefficients:* $g \in \mathbb{Z}[[t]]$, then $g(t)$ is also algebraic, $g(t) \in \mathbb{Z}[[t]] \cap \overline{\mathbb{Q}(t)}$.

Indeed, we have the 1-st order equation on series $g(t)$

$$t\partial_t g(t) = f(t)g(t)$$

which is a connection on the trivial line bundle on an algebraic curve.

Application to hypergeometric examples

Recently É.Delaygue and T.Rivoal (arXiv:2209.03301) proved the conjecture attributed by D.Zagier to V.Golyshev: any algebraic hypergeometric series of factorial type is the logarithmic derivative of an algebraic function.

For example,

$$\exp\left(\sum_{n\geqslant 1}\frac{1}{n}\frac{(6n)!n!}{(3n)!(2n)!(2n)!}t^n\right)=1+30t+1605t^2+107218t^3+8043114t^4+\dots$$

is an algebraic series with integer coefficients.

The integrality of coefficients is proven using the well-known Dwork criterion:

$$\exp\Big(\sum_{n\geqslant 1}h_n t^n\Big)\in\mathbb{Z}[[t]] \iff h_{pn}-\frac{1}{p}h_n\in\mathbb{Z}_p \quad \forall n\geqslant 1, \forall \text{ prime } p$$

The algebraicity is the direct application of the theorem of D.Chudnovsky and G.Chudnovsky.

Also, there is an extension to the case of algebraic hypergeometric series which are *not* of factorial type.

Application to free probability

Let A is the group ring of the free finitely-generated group:

$$A = \mathbb{C}[Free_k] = \mathbb{C}\langle u_1^{\pm 1}, \ldots, u_k^{\pm 1}\rangle$$

and the functional $\tau : A \to \mathbb{C}$ is given by

$$\tau : \sum_{g \in Free_k} c_g \cdot g \mapsto c_{id} \in \mathbb{C}$$

Theorem (M.Schützenberger, N.Chomsky (1963))

For any $a \in A$ the series

$$\tau((1 - t \cdot a)^{-1}) := \sum_{n \geqslant 0} \tau(a^n) t^n \in \mathbb{C}[[t]]$$

is algebraic.

This is a corollary of theory of algebraic series in free variables.

Definition

A noncommutative series $f \in \mathbb{C}\langle\langle x_1, \dots, x_n \rangle\rangle$ is *rational* iff there exist square matrices $T_0, T_1, T_2, \dots T_n \in Mat(N \times N, \mathbb{C})$ for some $N \geqslant 0$ such that

$$f = \mathrm{Tr}\left[T_0 \cdot (1 - T_1 x_1 - T_2 x_2 - \cdots - T_n x_n)^{-1}\right]$$

Definition

A noncommutative series $f = 1 + \cdots \in \mathbb{C}\langle\langle x_1, \dots, x_n \rangle\rangle$ is *algebraic* iff there exist a collection of series $f_1, \dots, f_m$ such that $f_1 = f$, and a collection of noncommutative polynomials

$$g_j \in \mathbb{C}\langle x_1, \dots x_n, y_1, \dots, y_m \rangle, \quad j = 1, \dots, m$$

such that each monomial in each g_j contains at least one $x_\bullet$, and such that

$$f_j = 1 + g_j(x_1, \dots, x_n, f_1, \dots, f_m) \quad \forall j = 1, \dots, m$$

In the case of a series f with the constant term $\neq 1$, we call f algebraic iff the series obtained by the replacement of the constant term by 1, i.e. $\tilde{f} := f - f(0) + 1$, is algebraic.

The problem of triviality of an algebraic noncommutative series is algorithmically unsolvable.

The crucial result by Schützenberger (1961) is that the noncommutative Hadamard product of a rational noncommutative series with an algebraic one is again algebraic (the analogous statement in the commutative setting is *not true*):

Theorem

Let $f_r, f_a \in \mathbb{C}\langle\langle x_1, \ldots, x_n \rangle\rangle$ are two noncommutative formal series, such that f_a is rational and f_r is algebraic. Then

$$\sum_{\textit{words in } x_1,\ldots,x_n} \mathrm{Coeff}_{\textit{word}}(f_r)\, \mathrm{Coeff}_{\textit{word}}(f_a) \cdot \textit{word} \in \mathbb{C}\langle\langle x_1, \ldots, x_n \rangle\rangle$$

is algebraic.

The application to the group ring $A = \mathbb{C}[Free_k] = \mathbb{C}\langle u_1^{\pm 1}, \ldots, u_k^{\pm 1} \rangle$ and the trace functional τ is based on

Lemma

Consider the free algebra generated by the alphabet $x_1, \ldots, x_k, x_1^*, \ldots, x_k^*$. Then the sum over all words such that after the replacement $x_i^* \rightsquigarrow x_i^{-1}$, $i = 1, \ldots, k$ we get the identity element of $Free_k$, is a *noncommutative algebraic series.*

An improvement of Schützenberger and Chomsky theorem, also based on the $rk = 1$ case of the Algebraicity Conjecture proven by Chudnovsky and Chudnovsky:

Theorem (MK, 2011, preprint "Noncommutative identities", arXiv 1109.2469)

For any $a \in A = \mathbb{C}[Free_k]$ *the series* $\exp(\sum_{n\geqslant 1} \frac{\tau(a^n)}{n} t^n)$ *is algebraic.*

Idea of the proof: assume that $a \in \mathbb{Z}\langle u_1^{\pm 1}, \ldots, u_k^{\pm 1}\rangle$.

I claim that $\exp(\sum_{n\geqslant 1} \frac{\tau(a^n)}{n} t^n)$ is also a series with integer coefficients. Indeed, if $a = \sum_{g\in Free_k} c_g \cdot g \in \mathbb{Z}[Free_k]$ then

$$\exp(-\sum_{n\geqslant 1} \frac{\tau(a^n)}{n} t^n) = \prod_{m\geqslant 1} \prod_{(g_1,\ldots,g_m)} (1 - c_{g_1} \ldots c_{g_m} t^m) \in \mathbb{Z}[[t]]$$

where for any $m \geqslant 1$ we take the product over all sequences of elements of $Free_k$ such that $g_1 \ldots g_m = id$ and the sequence $(g_1, \ldots, g_m)$ is *strictly* smaller than all its cyclic permutations, for the e.g. lexicographic order. The only property of group $Free_k$ used here is that it is torsion-free.

At that time ($\sim$2011) my “interpretation” of the series

$$\exp(-\sum_{n\geqslant 1} \frac{\tau(a^n)}{n} t^n)$$

was purely formal:

if we replace A by the algebra of matrices of a given finite size, and τ by the usual trace (satisfying $\tau(ab) = \tau(ba)\ \forall a, b$ as in the case $A = \mathbb{C}[Free_k]$), then

$$\exp(-\sum_{n\geqslant 1} \frac{\tau(a^n)}{n} t^n) = \det(1 - t \cdot a)$$

is a *polynomial* in t, (up to a trivial modification) the characteristic polynomial of a.

Up to now there is no alternative (e.g. a purely combinatorial) proof of the algebraicity of the series $\exp(-\sum_{n\geqslant 1} \frac{\tau(a^n)}{n} t^n)$. We do not have even an explicit bound on the degree of the minimal algebraic equation satisfied by this series.

Example: for $a = u_1 + u_1^{-1} + \cdots + u_k + u_k^{-1}$ one has

$$\tau\Big(\frac{1}{1-ta}\Big) = \frac{2k-1}{k-1+k\sqrt{1-4(2k-1)t^2}}$$

$$\exp(\tau(\log(1-ta))) = \left[\frac{2k-1}{k-1+k\sqrt{1-4(2k-1)t^2}}\right]^{k-1} \cdot \left[\frac{1+\sqrt{1-4(2k-1)t^2}}{2}\right]^{k}$$

$\rightsquigarrow$ **Questions**: consider algebra A_E depending on parameter $E \in \mathbb{C}$ and giving by adding to $A = \mathbb{C}\langle u_1^{\pm 1}, \dots, u_k^{\pm 1}\rangle$ the two-sided inverse

$$(u_1 + u_1^{-1} + \cdots + u_k + u_k^{-1} - E)^{-1}$$

Does $HP(A_E)$ (periodic cyclic homology) jump at special values $E = \pm 2\sqrt{2k-1}$? (true for $k = 1$)

Maybe in general, the bundle $HP(A_E)$ on $\mathbb{C}$ with coordinate E, where $A_E = A(a-E)^{-1}$ has finite rank and *finite* monodromy?

Does the functional τ on $\mathbb{C}[Free_k]$ have any natural noncommutative meaning?

In certain sense, the answer is yes, it comes from free probability theory:

Theorem (D.Voiculescu)

Let $d\mu_N(\mathbf{u})$ be the Haar measure on the unitary group $U(N)$ with the total mass 1 (probability measure). Then for any word g in $u_1^{\pm 1}, \dots u_k^{\pm 1}$ the limit as $N \to +\infty$ of the mean value

$$\int_{U(N)^k} \frac{1}{N} \operatorname{Tr} g(\mathbf{u}_1, \dots, \mathbf{u}_k)\ d\mu_N(\mathbf{u}_1) \dots d\mu_N(\mathbf{u}_k)$$

is 1 if g maps to $id \in Free_k$, and 0 otherwise.

Notice that A is a $\star$-algebra, via $u_i^\dagger := u_i^{-1}$.

Corollary

For any self-adjoint element $a = a^\dagger \in A$ there exists a unique probability measure $d\mu_a$ on $\mathbb{R}$ with compact support such that for any $n \in \mathbb{Z}_{\geqslant 0}$ one has

$$\tau(a^n) = \int_{\mathbb{R}} x^n d\mu_a(x)$$

Proof.

Let us associate with a hermitian matrix X of size $N \times N$ a probability measure on $\mathbb{R}$ given by

$$d\mu_X(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i)$$

where $(\lambda_i \in \mathbb{R})_{1 \leqslant i \leqslant N}$ are eigenvalues of X. Then $\tau(a^n)$ is the limit as $N \to \infty$ of the n-th moment of random probability measure $d\mu_X$ where $X = a(\mathbf{u}_1, \ldots, \mathbf{u}_k)$. □

Moreover, it is easy to see that for each $n \geqslant 0$ the random variable $\frac{1}{N} \operatorname{Tr} a(\mathbf{u}_1, \ldots, \mathbf{u}_k)^n$ depending on size N of matrices, tends to a *non-random constant* as $N \to \infty$. Therefore, *random* probability measure $d\mu_X$ tends as $N \to \infty$ to certain *definite* probability measure on $\mathbb{R}$ with compact support, with the n-th moment equal to $\tau(a^n)$:

$$\tau(a^n) = \int x^n d\mu_a(x) \quad \forall\, n \geqslant 0$$

Algebracity results above can be rephrased as

- $\int \frac{1}{1-tx}\, d\mu_a(x)$ is algebraic for $|t| \ll 1$
- $\exp\left(\int \log(1-tx)\, d\mu_a(x)\right)$ is algebraic for $|t| \ll 1$

Corollary:

- *the density* $\rho(x) = \frac{d\mu_a(x)}{dx}$ *is a piece-wise algebraic function of* $x \in \mathbb{R}$

$\Leftarrow$ Stieltjes-Perron formula: $\rho(x) = \frac{S_\rho(x-\mathrm{i}0)-S_\rho(x+\mathrm{i}0)}{2\pi\mathrm{i}}$ where $S_\rho(y) := \int \frac{\rho(x)}{y-x}dx,\ y \notin \mathbb{R}$.

Example:

$$d\mu_{u_1+u_1^{-1}+\cdots+u_k+u_k^{-1}}(x) = \frac{1}{\pi}\frac{k\sqrt{4(2k-1)-x^2}}{4k^2-x^2}dx, \quad x \in [-2\sqrt{2k-1}, +2\sqrt{2k-1}]$$

GUE version

Recall another probability measure on matrices, called Gaussian Unitary Ensemble:

$$d\mu_{N,GUE}(X) = \frac{1}{Z_N} e^{-N\frac{\operatorname{Tr} X^2}{2}} d^{N^2}X, \qquad Z_N := \left(\frac{\sqrt{2\pi}}{\sqrt{N}}\right)^{N^2}$$

where X is a Hermitian $N \times N$ matrix.

Let now A be free algebra $\mathbb{C}\langle x_1, \ldots, x_k\rangle$ with $\star$-structure $x_i^\dagger = x_i$. Define a trace

$$\tau : A \to \mathbb{C}, \qquad \tau(word) := \# \overset{\iff \nexists \ \ldots\bullet\ldots\bullet\ldots\bullet\ldots\bullet\ldots}{\text{non-crossing pairings for}} \ word$$

Example: $\tau(x_1x_1x_1x_1x_2x_3x_3x_2) = 2$: 11112332 and 11112332

Similarly with the unitary story, for any self-adjoint element $a = a^\dagger \in A$ one gets the limiting distribution $d\mu_a(x)$ of eigenvalues of random (non-Gaussian) Hermitian matrix $a(\mathbf{x}_1, \ldots, \mathbf{x}_k)$, with the moments given by

$$\tau(a^n) = \int x^n d\mu_a(x) \quad \forall\, n \geqslant 0$$

We have a theorem as before: $\int \frac{1}{1-tx} d\mu_a(x)$ is algebraic for $|t| \ll 1$ (but not the exponential generalization!):

use the fact that the noncommutative series

$$P := \sum_{words \;\; in \;\; x_1,\ldots,x_n} \tau(word) \cdot word \in \mathbb{Z}\langle\langle x_1, \ldots, x_n \rangle\rangle$$

is algebraic, and Schützenberger's result on Hadamard products.

Indeed, P is the unique solution of the quadratic equation

$$P = 1 + \sum_i x_i \, P \, x_i \, P$$

(alternatively, P is the unique solution of the equation $P = 1 + \sum_i P \, x_i \, P \, x_i$).

Again, using Stieltjes-Perron formula, we get

Corollary: *the density* $\frac{d\mu_a(x)}{dx}$ *is a piece-wise algebraic function of* $x \in \mathbb{R}$

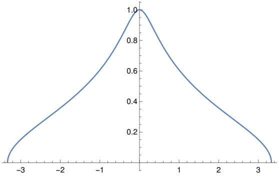
Examples:

1. $k = 1, a = x_1 \implies g(t) := \tau\left(\frac{1}{1-ta}\right)$ satisfies $g = 1 + t^2 g^2$ and the density is Wigner semi-circle law supported on $[-2, 2] \subset \mathbb{R}$:

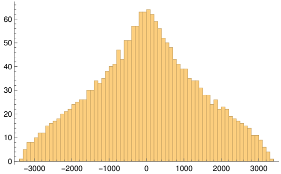
$$d\mu_a(x) = \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}\,dx$$

2. $k = 2, a = x_1x_2 + x_2x_1 \implies g(t) := \tau\left(\frac{1}{1-ta}\right)$ satisfies $g = 1 + t^2(g^2 + g^3)$ and the density is supported on $[-x_0, x_0] \subset \mathbb{R}$ where $x_0 = \sqrt{\frac{11+5\sqrt{5}}{2}}, \quad 1 + 11x_0^2 = x_0^4$

$$d\mu_a(x) = \frac{1}{\pi}V(x)dx, \quad 1 + 11x^2 - x^4 = V^2(1 + 3x^2 + 4V^2x^2)^2$$



graph of the density $V(x)$



histogram of eigenvalues of $X_1X_2 + X_2X_1$ for random symmetric 2000×2000 matrices

Let us recapitulate: we have 3 (in the case of unitary matrices) or 2 (in the case of GUE) algebraic functions:

- the limiting density of eigenvalues $\rho(x) = \frac{d\mu_a(x)}{dx}$
- its Cauchy-Stieltjes transform $S_\rho(y) := \int \frac{\rho(x)}{y-x} dx, \ y \notin \mathbb{R}$
- (*in the unitary case*): function $\exp(\int \log(y-x)\rho(x)dx), \ y \notin Supp(\rho)$

What is the matrix-theoretic meaning of the integral $\int \log(y-x)\rho(x)dx$?

Surprisingly, such an integral for $y \in Supp(\rho)$ appears naturally in the Hermitian one-matrix models (a generalization of GUE for one matrix):
consider the probability measure on the space of Hermitian $N \times N$ matrices given by

$$\frac{1}{Z_N} e^{-N \operatorname{Tr} W(X)} d^{N^2} X, \quad X^\dagger = X \in Mat(N \times N, \mathbb{C})$$

where $W \in \mathbb{R}[x]$ is a polynomial of even degree such that $W(x) \to +\infty$ as $x \to \pm\infty$.

Then the limiting density of eigenvalues of X in the planar limit $N \to \infty$ satisfies Euler-Lagrange equation:

$$\delta_{\rho: \int \rho(x)dx=1} \left[\int \int \rho(x)\rho(y) \log|x-y| dx\, dy - \int W(x)\rho(x)dx \right] = 0$$

$$\Longrightarrow \boxed{2 \int_y \log|x-y|\ \rho(y)dy = const + W(x)} \text{ on the support of } \rho$$

where $const \in \mathbb{R}$ - is unknown a priori Lagrange multiplier (for the constraint $\int \rho(x)dx = 1$).

Corollary: $2 \cdot v.p. \int \frac{\rho(y)}{y-x} dy = W'(x) \iff S(x + \mathrm{i} \cdot 0) + S(x - \mathrm{i} \cdot 0) = W'(x)$

Examples:

- $\rho_{GUE} = \frac{1}{2\pi}\sqrt{4-x^2} \quad x \in [-2,2], \; W(x) = \frac{x^2}{2}$

  $2\int_{-2}^{2} \log|y-x| \cdot \frac{1}{2\pi}\sqrt{4-x^2}dy = \frac{x^2}{2} - 1 \quad \forall x \in [-2,2]$

- $\rho_{u_1+u_1^{-1}+\cdots+u_k+u_k^{-1}} = \frac{1}{\pi}\frac{k\sqrt{4(2k-1)-x^2}}{4k^2-x^2} \quad x \in [a,b] := [-2\sqrt{2k-1}, +2\sqrt{2k-1}]$

  special case $k=1$: $\rho(x) = \frac{1}{\pi}\frac{1}{\sqrt{4-x^2}}, \quad x \in [-2,2]$

  $2\int \log|x-y|\rho(y)dy = \log\frac{(2k-1)^{2k-1}}{(4k^2-x^2)^{k-1}} \quad \forall y \in [a,b]$

  special case $k=1$: $2\int \log|x-y|\rho(y)dy = 0$

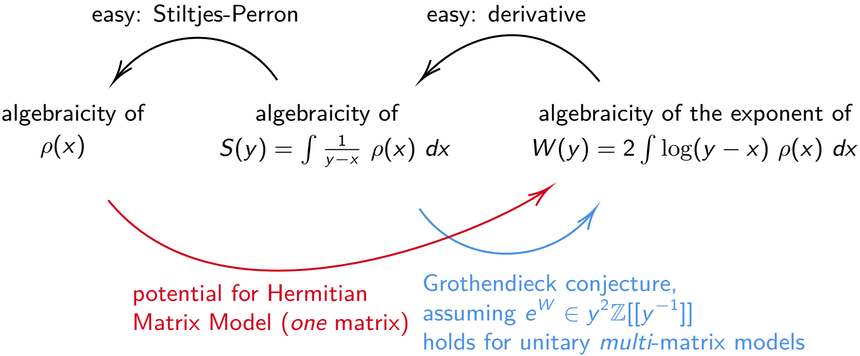
The last equality gives an amazing coincidence of limiting eigenvalue distributions: $\mathbf{u}_1 + \mathbf{u}_1^{-1}$ where $\mathbf{u}_1$ is a random unitary matrix, has the *same* distribution of eigenvalues as a random Hermitian matrix X_1 with the Lebesgue measure $d^{N^2}X$ satisfying the constraint

$$|X_1| \leq 2$$

where $|X|$ is the *operator norm* (in other words, all eigenvalues of X lie in $[-2,2]$).

Conclusion: Algebraicity of $\exp(\sum_{n\geqslant 1} \frac{\tau(a^n)}{n} t^n)$ for the case of unitary matrices, $a = a^\dagger \in \mathbb{C}[Free_k]$ implies that $X = a(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ has the *same* distribution of eigenvalues in the planar limit $N \to \infty$ as the matrix in the Hermitian matrix model (for only one matrix) with the non-polynomial potential

$$W(x) = \log(\text{piece-wise algebraic function of } x)$$


easy: Stiltjes-Perron
easy: derivative
algebraicity of $\rho(x)$
algebraicity of $S(y) = \int \frac{1}{y-x}\ \rho(x)\ dx$
algebraicity of the exponent of $W(y) = 2\int \log(y-x)\ \rho(x)\ dx$
potential for Hermitian Matrix Model (*one* matrix)
Grothendieck conjecture, assuming $e^W \in y^2\mathbb{Z}[[y^{-1}]]$ holds for unitary *multi*-matrix models