Picard–Fuchs operators for Feynman integrals : algorithms and motives

Pierre Vanhove

IPhT Saclay

Séminaire “Équations différentielles motiviques et au–delà”

based on 2209.10962 with Pierre Lairez and work to appear with
Charles Doran, Andrew Harder, Eric Pichon

Feynman Integrals

Any Feynman integrals has the parametric representation

$$I_\Gamma(\underline{s}, \underline{m}; \underline{\nu}, D) = \int_{\Delta_n} \Omega_\Gamma; \quad \Omega_\Gamma := \frac{\mathcal{U}_\Gamma(\underline{x})^{\nu - \frac{(L+1)D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^{\nu - \frac{LD}{2}}} \prod_{i=1}^{n} x_i^{\nu_i - 1}\, dx_i$$

with $\nu = \sum_{i=1}^{n} \nu_i$, $L \in \mathbb{N}$, $D \in \mathbb{C}$, $\nu_i \in \mathbb{Z}$

The domain of integration is the positive quadrant

$$\Delta_n := \{x_1 \geqslant 0, \ldots, x_n \geqslant 0 | [x_1, \ldots, x_n] \in \mathbb{P}^{n-1}\}$$

The graph polynomial is homogeneous degree $L+1$ in $\mathbb{P}^{n-1}$

$$\mathcal{F}_\Gamma(\underline{x}) = \mathcal{U}_\Gamma(\underline{x}) \times L(\underline{m}^2; \underline{x}) - \mathcal{V}_\Gamma(\underline{s}, \underline{x})$$

Feynman graph polynomials

The graph polynomial is homogeneous degree $L+1$ in $\mathbb{P}^{n-1}$

$$\mathcal{F}_\Gamma(\underline{x}) = \mathcal{U}_\Gamma(\underline{x}) \times L(\underline{m}^2; \underline{x}) - \mathcal{V}_\Gamma(\underline{s}, \underline{x})$$

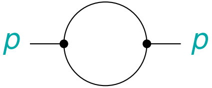
- Homogeneous polynomial of degree L with $u_{a_1,\dots,a_n} \in \{0,1\}$

$$\mathcal{U}_\Gamma(\underline{x}) = \sum_{\substack{a_1+\cdots+a_n=L \\ 0\leqslant a_i \leqslant 1}} u_{a_1,\dots,a_n} \prod_{i=1}^n x_i^{a_i}$$

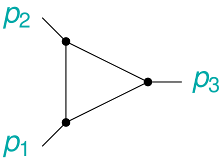
- the mass hyperplane $L(\underline{m}^2; \underline{x}) := \sum_{i=1}^n m_i^2 x_i$
- Homogeneous polynomial of degree $L+1$ with $s_{a_i,\cdots,a_n}$ are linear combination of the product of the external momenta $\underline{s} = \{p_i \cdot p_j\}$

$$\mathcal{V}_\Gamma(\underline{x}) = \sum_{\substack{a_1+\cdots+a_n=L+1 \\ 0\leqslant a_i \leqslant 1}} s_{a_i,\cdots,a_n} \prod_{i=1}^n x_i^{a_i}$$

Feynman graph polynomials : examples

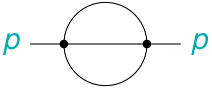


$$\begin{cases} U_\circ = x_1 + x_2 \\ V_\circ = p^2 x_1 x_2 \end{cases}$$



$$\begin{cases} U_\triangleright = x_1 + x_2 + x_3 \\ V_\triangleright = p_1^2 x_2 x_3 + p_2^2 x_1 x_3 + p_3^2 x_1 x_2 \end{cases}$$

with $p_1 + p_2 + p_3 = 0$



$$\begin{cases} U_\ominus = x_1 x_2 + x_1 x_3 + x_2 x_3 \\ V_\circ = p^2 x_1 x_2 x_3 \end{cases}$$

Noboru Nakanishi
Graph theory and Feynman integrals
New York : Gordon and Breach, (1971)

Feynman Integrals: divergences

As a function of the powers of the propagators $\underline{\nu}$ and the dimension D the integral has singularities located on hyperplane defined by $\sum_{i=1}^{n} a_i\nu_i + a_0 D = 0$ with $(a_0, a_1, \dots, a_n) \in \mathbb{Z}^{n+1}$
One can perform a Laurent expansion near say $D_c = 4$ dimensions

$$I_\Gamma(\underline{s}, \underline{m}^2; \underline{\nu}, D) = \sum_{r\geqslant -2L} (D - D_c)^r I_\Gamma^{(r)}(\underline{s}, \underline{m}^2; \underline{\nu})$$

where $I_\Gamma^{(r)}(\underline{s}, \underline{m}^2; \underline{\nu})$ are convergent integrals.
From now we will consider convergent integrals

Eugene R. Speer
Generalized Feynman Amplitudes
Princeton University Press, (1969)

G.'t Hooft, M.Veltman
Regularization and renormalization of gauge fields
*Nucl. Phys. B **44** (1972), 189-213*

Feynman Integrals: motivic period integrals

$$I_\Gamma(\underline{s},\underline{m};\underline{\nu},D)=\int_{\Delta_n}\Omega_\Gamma,\qquad \Omega_\Gamma=\frac{\mathcal{U}_\Gamma(\underline{x})^{\omega-\frac{D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^{\omega}}\prod_{i=1}^{n}x_i^{\nu_i-1}\,\Omega_0$$

with $\omega=\sum_{i=1}^n\nu_i-\frac{LD}{2}$ and the volume form on $\mathbb{P}^{n-1}$

$$\Omega_0=\sum_{i=1}^{n}(-1)^{i-1}x^i dx^1\wedge\cdots\widehat{dx^i}\cdots\wedge dx^n$$

The integrand is an algebraic differential form in $H^{n-1}(\mathbb{P}^{n-1}\backslash Z_\Gamma)$ on the complement of the graph hypersurface Z_Γ determined by the vanishing locus of the denominator

- All the singularities of the Feynman integrals are located on the graph hypersurface
- Generically the graph hypersurface has non-isolated singularities

Feynman Integrals: Picard-Fuchs equation

The modern interpretation is that Feynman integrals are (motivic) periods of the algebraic geometry defined by the graph hypersurface

A Feynman integral satisfies inhomogenous differential equations with respect to any set of variables $\underline{z} \in \{p_i \cdot p_j, m_1^2, \ldots, m_n^2\}$

$$\mathcal{L}_\Gamma(\underline{z})\, I_\Gamma = \mathcal{S}_\Gamma$$

The aim of this talk is to present some new methods for deriving such system of differential equation and its underlying (algebraic) geometry

Feynman Integrals are D-finite functions

Theorem [Kashiwara, Kawai ;Petukhov, Smirnov; Bitoun et al.]
Feynman integrals are **holonomic D-finite functions** for generic values of $\underline{\nu}_i$ and D

Definition A multivariate function $f(x_1, \dots, x_n)$ is **D-finite** over $\mathbb{F}(x_1, \dots, x_n)$ if for each $i \in \{1, \dots, n\}$ the function satisfies a linear partial differential equation

$$c_{i,r_i}(\underline{x}) \frac{\partial^{r_i}}{\partial x_i^{r_i}} f(\underline{x}) + c_{i,r_i-1}(\underline{x}) \frac{\partial^{r_i-1}}{\partial x_i^{r_i-1}} f(\underline{x}) + \cdots + c_{i,0}(\underline{x}) = 0 \,.$$

where the coefficients $c_{i,j}(\underline{x}) \in \mathbb{F}[x_1, \dots, x_n]$.

Feynman Integrals differential equations

This is a consequence of the fact that the dimension of the vector space is finite [Bitoun, Bogner, Klausen, Panzer]

$$V_\Gamma := \sum_{\underline{\nu}\in\mathbb{Z}^n} \mathbb{C}(-D/2,\underline{\nu}) I_\Gamma(\underline{s},\underline{m}^2;\underline{\nu},D)$$

The dimension of this space is given by the Euler characteristic of the complement of the graph hyper-surface

$$\dim(V_\Gamma) = (-1)^{n+1}\chi\left((\mathbb{C}^*)^n \backslash \mathbb{V}(\mathcal{U}_\Gamma)\cup\mathbb{V}(\mathcal{F}_\Gamma)\right),$$

with the vanishing loci defined

$$\mathbb{V}(\mathcal{U}_\Gamma) := \{\mathcal{U}_\Gamma(\underline{x}) = 0 | \underline{x}\in\mathbb{P}^{n-1}\}$$

$$\mathbb{V}(\mathcal{F}_\Gamma) := \{\mathcal{F}_\Gamma(\underline{x}) = 0 | \underline{x}\in\mathbb{P}^{n-1}\}$$

Feynman Integrals differential equations

Let $M_\Gamma(\underline{s})$ be a basis of the space V_Γ for the family of integral $I_\Gamma(\underline{s}, \underline{m}^2; \underline{\nu}, D)$

Differentiating with respect to the physical parameters $\underline{s}$, the elements of $dM_\Gamma(\underline{s})$ are integrals in the family $I_\Gamma(\underline{s}, \underline{m}^2; \underline{\nu}, D)$ with different indices $\underline{\nu}$. This is because the V_Γ graph polynomial is a linear combination of kinematic variables.

Therefore we have the first order differential system of equations

$$dM_\Gamma(\underline{s}) = A_\Gamma(\underline{s}) \wedge M_\Gamma(\underline{s})$$

The matrix A_Γ is a flat connection satisfying

$$dA_\Gamma(\underline{s}) + A_\Gamma(\underline{s}) \wedge A_\Gamma(\underline{s}) = 0$$

Feynman Integrals differential equations

One can convert the first order system into a system of differential operators acting on the the Feynman integral $I_\Gamma(\underline{s}, \underline{m}^2; \underline{\nu}, D)$.

$$dM_\Gamma(\underline{s}) = A_\Gamma(\underline{s}) \wedge M_\Gamma(\underline{s}).$$

For the case of the two-loop sunset in $D = 2$

$$\Omega_\ominus(p^2) = \frac{\Omega_0}{\mathcal{F}_\ominus(x_1, x_2, x_3)}, \quad \mathcal{F}_\ominus(x_1, x_2, x_3) = x_1 x_2 x_3 \left(\left(\sum_{i=1}^{3} \frac{1}{x_i} \right) \left(\sum_{j=1}^{3} m_j^2 x_j \right) - p^2 \right)$$

$\dim(V_\ominus) = 7$ but $\mathrm{rank}(A_\ominus(p^2)) = 2$ because $\mathcal{F}_\ominus = 0$ defines an elliptic curve

The main goal of this work is to identify the rank of the associated Hodge structure (motive)

Feynman Integrals differential equations

For a given subset of the physical parameters $\underline{z} := (z_1, \ldots, z_r) \subset \{\underline{s}, \underline{m}^2\}$ we want to derive **minimal order** differential equations

$$\mathcal{L}_\Gamma(\underline{s}, \underline{m}^2, \partial_{\underline{z}}) \int_\sigma \frac{\mathcal{U}_\Gamma(\underline{x})^{\omega - \frac{D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^\omega} \prod_{i=1}^n x_i^{\nu_i - 1}\, \Omega_0 = \mathcal{S}_{\sigma,\Gamma}(\underline{z})$$

One way to achieve this is to construct a Gröbner basis of operators $T_{\underline{z}}$ that annihilate the integrand of the Feynman integral

$$T_{\underline{z}} \left(\frac{\mathcal{U}_\Gamma(\underline{x})^{\omega - \frac{D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^\omega} \prod_{i=1}^n x_i^{\nu_i - 1}\, \Omega_0 \right) = 0$$

such that

$$T_{\underline{z}} = \mathcal{L}_\Gamma(\underline{s}, \underline{m}^2, \underline{\partial}_{\underline{z}}) + \sum_{i=1}^n \partial_{x_i} Q_i(\underline{s}, \underline{m}^2, \underline{\partial}_{\underline{z}}; \underline{x}, \underline{\partial}_{\underline{x}})$$

Feynman Integrals differential equations

where the finite order differential operator

$$\mathcal{L}_\Gamma(\underline{s}, \underline{m}^2, \underline{\partial_z}) = \sum_{\substack{0\leqslant a_i \leqslant o_i\\ 1\leqslant i\leqslant r}} p_{a_1,\dots,a_r}(\underline{s}, \underline{m}^2) \prod_{i=1}^{r} \left(\frac{d}{dz_i}\right)^{a_i}$$

$$Q_i(\underline{s}, \underline{m}^2, \underline{\partial_z}) = \sum_{\substack{0\leqslant a_i \leqslant o'_i\\ 1\leqslant i\leqslant r}} \sum_{\substack{0\leqslant b_i \leqslant \tilde{o}_i\\ 1\leqslant i\leqslant n}} q^{(i)}_{a_1,\dots,a_r}(\underline{s}, \underline{m}^2, \underline{x}) \prod_{i=1}^{r} \left(\frac{d}{dz_i}\right)^{a_i} \prod_{i=1}^{n} \left(\frac{d}{dx_i}\right)^{b_i}$$

- The orders o_i, o'_i, $\tilde{o}_i$ are positive integers
- $p_{a_1,\dots,a_r}(\underline{S}, \underline{m}^2)$ polynomials in the kinematic variables
- $q^{(i)}_{a_1,\dots,a_r}(\underline{s}, \underline{m}^2, \underline{x})$ rational functions in the kinematic variable and the projective variables $\underline{x}$.

Feynman Integrals differential equations

Integrating over a cycle γ gives

$$0 = \oint_\gamma T_{\underline{z}}\Omega_\Gamma = \mathcal{L}_\Gamma(\underline{s}, \underline{m}, \partial_{\underline{z}}) \oint_\gamma \Omega_\Gamma + \oint_\gamma d\beta_\Gamma$$

For a cycle $\partial\gamma = \emptyset$ then $\oint_\gamma d\beta_\Gamma = 0$ and we get

$$\mathcal{L}_\Gamma(\underline{s}, \underline{m}, \partial_{\underline{z}}) \oint_\gamma \Omega_\Gamma = 0$$

For the Feynman integral I_Γ we have

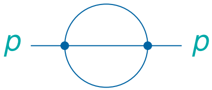
$$0 = \int_{\Delta_n} T_{\underline{z}}\Omega_\Gamma = \mathcal{L}_\Gamma(\underline{s}, \underline{m}, \partial_{\underline{z}}) I_\Gamma + \int_{\Delta_n} d\beta_\Gamma$$

since $\partial\Delta_n \neq \emptyset$

$$\mathcal{L}_\Gamma(\underline{s}, \underline{m}, \partial_{\underline{z}}) I_\Gamma = \mathcal{S}_\Gamma$$

The sunset graph

The two-loop sunset graph in $D = 2$



$$I_\ominus(p^2, \underline{m}^2) = \int_{\mathbb{R}_+^3} \frac{dx_1 dx_2 dx_3}{\mathcal{F}_\ominus(\underline{x})}$$

The polar hypersurface of the integral is an elliptic curve $\mathcal{F}_\ominus(\underline{x}) = 0$

$$\mathcal{F}_\ominus(\underline{x}) = (x_1 x_2 + x_1 x_3 + x_2 x_3)(m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3) - p^2 x_1 x_2 x_3$$

The sunset graph : Griffiths-Dwork method

One can obtain a differential equation annihilating acting on the integral using the Griffiths-Dwork method

Let define the integrand in differential form

$$\eta_\ominus = \frac{x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2}{\mathcal{F}_\ominus(\underline{x})} = \frac{\Omega_0}{\mathcal{F}_\ominus(\underline{x})}$$

consider

$$\frac{\partial \eta_\ominus}{\partial p^2} = x_1 x_2 x_3 \frac{\Omega_0}{\mathcal{F}_\ominus(\underline{x})^2}; \qquad \frac{\partial^2 \eta_\ominus}{(\partial p^2)^2} = 2(x_1 x_2 x_3)^2 \frac{\Omega_0}{\mathcal{F}_\ominus(\underline{x})^3}$$

Since we know we have the geometry of an elliptic curve we are looking for a second order differential operator acting on $\eta_\ominus$

$$\mathcal{L}_\ominus(p^2) = \frac{\partial^2}{(\partial p^2)^2} + q_1(p^2, \underline{m}^2)\frac{\partial}{\partial p^2} + q_0(p^2, \underline{m}^2)$$

The sunset graph : Griffiths-Dwork method

Remark that $(x_1 x_2 x_3)^2$ lies in the Jacobian ideal for $\mathcal{F}_\ominus(\underline{x})$

$$(x_1 x_2 x_3)^2 = \sum_{i=1}^{3} C_i^{(1)}(\underline{x}) \partial_{x_i} \mathcal{F}_\ominus(\underline{x})$$

with $C_i^{(1)}(\underline{x})$ homogeneous of degree 4 in the (x_1, x_2, x_3) variables
Following Griffiths one introduces the differential form

$$\beta_1 = \frac{(x_2 C_3^{(1)}(\underline{x}) - x_3 C_2^{(1)}(\underline{x}))dx_1}{\mathcal{F}_\ominus(\underline{x})^2} + \frac{(x_1 C_3^{(1)}(\underline{x}) - x_3 C_1^{(1)}(\underline{x}))dx_2}{\mathcal{F}_\ominus(\underline{x})^2} + \frac{(x_1 C_2^{(1)}(\underline{x}) - x_2 C_1^{(1)}(\underline{x}))dx_3}{\mathcal{F}_\ominus(\underline{x})^2}$$

such that

$$d\beta_1 = 2\frac{\sum_{i=1}^3 C_i^{(1)}(\underline{x}) \partial_{x_i} \mathcal{F}_\ominus(\underline{x}) \Omega_0}{\mathcal{F}_\ominus(\underline{x})^3} - \frac{\sum_{i=1}^3 \partial_{x_i} C_i^{(1)}(\underline{x}) \Omega_0}{\mathcal{F}_\ominus(\underline{x})^2}$$

The sunset graph : Griffiths-Dwork method

$$\mathcal{L}_{\ominus}(p^2)\eta_{\ominus} = \frac{q_1(p^2,\underline{m}^2)x_1x_2x_3 + \sum_{i=1}^{3}\partial_{x_i}C_i^{(1)}(\underline{x})}{\mathcal{F}_{\ominus}(\underline{x})^2}\Omega_0 + d\beta_1$$

We can again reduce this second order pole using that there exist a polynomial $q_1(p^2,\underline{m}^2)$ such that

$$q_1(p^2,\underline{m}^2)x_1x_2x_3 + \sum_{i=1}^{3}\partial_{x_i}C_i^{(1)}(\underline{x}) = \sum_{i=1}^{3}C_i^{(2)}\partial_{x_i}\mathcal{F}_{\ominus}(\underline{x})$$

with $C_i^{(2)}$ of degree 1. One introduces the 1-form β_2

$$\beta_2 = \sum_{i=1}^{3}\epsilon^{ijk}\frac{x_jC_k^{(2)}(\underline{x})dx_i}{\mathcal{F}_{\ominus}(\underline{x})}$$

such that

$$d\beta_2 = \frac{\sum_{i=1}^{3} C_i^{(2)}(\underline{x})\partial_{x_i}\mathcal{F}_{\ominus}(\underline{x})}{\mathcal{F}_{\ominus}(\underline{x})^2} - \frac{\sum_{i=1}^{3} \partial_{x_i} C_i^{(2)}(\underline{x})\Omega_0}{\mathcal{F}_{\ominus}(\underline{x})}$$

We have achieved that

$$\left(\frac{\partial^2}{(\partial p^2)^2} + q_1(p^2,\underline{m}^2)\frac{\partial}{\partial p^2} + \sum_{i=1}^{3}\partial_{x_i} C_i^{(2)}(\underline{x})\right)\eta_{\ominus} = d(\beta_1+\beta_2)$$

because the $C_i^{(2)}(\underline{x})$ are of degree 1 in (x_1, x_2, x_3) then $q_0(p^2,\underline{m}^2) = \partial_{x_i} C_i^{(2)}(\underline{x})$ only depends on $p^2, \underline{m}^2$

The sunset graph : Griffiths-Dwork method

We then conclude that the minimal operator acting on the sunset integral is the Picard-Fuchs operator

$$\mathcal{L}_{p^2} = \frac{\partial^2}{(\partial p^2)^2} + q_1(p^2, \underline{m}^2)\frac{\partial}{\partial p^2} + q_0(p^2, \underline{m}^2)$$

which acts on the integrals as

$$\mathcal{L}_{p^2} I_{\ominus}(p^2) = \int_{x_i \geqslant 0} \mathcal{L}_{p^2} \frac{\Omega_0}{\mathcal{F}_{\ominus}(\underline{x})} = \int_{x_i \geqslant 0} d(\beta_1 + \beta_2) \neq 0$$

We have constructed by the telescoper $T_{p^2} = \mathcal{L}_{\ominus}(p^2)$ and the certificate $C_{\ominus} = d(\beta_1 + \beta_2)$
The differential operator $\mathcal{L}_{p^2}$ is the Picard–Fuchs operator of the elliptic curve defined by the graph polynomial $\mathcal{F}_{\ominus}(x_1, x_2, x_3) = 0$

The sunset graph : Griffiths-Dwork method

If one considers the family of elliptic curve E

$$y^2 = 4x^3 - g_2(t)x - g_3(t); \qquad j(t) = \frac{g_2(t)^3}{\Delta(t)}$$

the periods satisfy the differential system of equations

$$\frac{d}{dt}\begin{pmatrix} \int_\gamma \frac{dx}{y} \\ \int_\gamma \frac{xdx}{y} \end{pmatrix} = \begin{pmatrix} -\frac{1}{12}\frac{d}{dt}\log\Delta(t) & \frac{3\delta(t)}{2\Delta(t)} \\ -\frac{g_2(t)\delta(t)}{8\Delta(t)} & \frac{1}{12}\frac{d}{dt}\log\Delta(t) \end{pmatrix}\begin{pmatrix} \int_\gamma \frac{dx}{y} \\ \int_\gamma \frac{xdx}{y} \end{pmatrix}$$

with $\delta(t) = 3g_3(t)\frac{d}{dt}g_2(t) - 2g_2(t)\frac{d}{dt}g_3(t)$

The Picard–Fuchs operator acting on the period integral $\int_\gamma dx/y$ is

$$\mathcal{L}_{\rm ell} = 144\Delta(t)^2\delta(t)\frac{d^2}{dt^2} + 144\Delta(t)\left(\delta(t)\frac{d\Delta(t)}{dt} - \Delta(t)\frac{d\delta(t)}{dt}\right)\frac{d}{dt}$$
$$+ 27g_2(t)\delta(t)^3 + 12\frac{d^2\Delta(t)}{dt^2}\delta(t)\,\Delta(t) - \left(\frac{d\Delta(t)}{dt}\right)^2\delta(t) - 12\frac{d\delta(t)}{dt}\Delta(t)\,\frac{d\Delta(t)}{dt}.$$

This matches the differential operator derived using the Griffiths–Dwork method

Extended Griffiths-Dwork algorithms

In general the graph hypersurface does not have isolated singularities (which is the generic case) therefore the “naïve” implementation of the Griffiths-Dwork algorithm does not work

One could use the implementation of Doron Zeilberger (1990) creative telescoping algorithm by F. Chyzak or K. Koutschan but the algorithm takes a very long time for graph with many edges

We will then use the implementation by Pierre Lairez of an extended Griffiths-Dwork algorithm that handles singular hypersurfaces

Extended Griffiths-Dwork: syzygies

In this example we saw the pole reduction

$$\frac{\partial^2\eta_\ominus}{(\partial p^2)^2} = 2(x_1x_2x_3)^2\frac{\Omega_0}{\mathcal{F}_\ominus(\underline{x})^3}\Omega_0 = \frac{\sum_{i=1}^3 \partial_{x_i}C_i}{\mathcal{F}_\ominus(\underline{x})^2}\Omega_0 + d\beta_1$$

For singular hypersurface $X_\Gamma \subset \mathbb{P}^{n-1}$ the Jacobian reduction may not be enough to reduce the pole order when $k \geqslant n$

Other reduction rules come from the *syzygies* of the derivatives $\frac{\partial\mathcal{F}_\Gamma}{\partial x_i}$, i.e. tuples $(B_1,\dots,B_n)$ be homogeneous of degree $k\deg\mathcal{F}_\Gamma - n + 1$ such that $\sum_i B_i\frac{\partial\mathcal{F}_\Gamma}{\partial x_i} = 0$ such that ($\xi_i = (-1)^{i-1}dx_1\cdots\widehat{dx_i}\cdots dx_n$)

$$\frac{\sum_i \frac{\partial B_i}{\partial x_i}}{\mathcal{F}_\Gamma^k}\Omega_0 = d\left(\sum_i \frac{B_i}{\mathcal{F}_\Gamma^k}\xi_i\right) \Longrightarrow \int_\gamma \frac{\sum_i \frac{\partial B_i}{\partial x_i}}{\mathcal{F}_\Gamma^k}\Omega_0 = 0.$$

In singular cases, these relations are missed by the Griffiths–Dwork reduction, we need the extended Griffiths–Dwork reduction implemented by [Lairez]

Extended Griffiths-Dwork : syzygies

Given a form $\Omega = \frac{A}{\mathcal{F}_\Gamma^k} d\underline{x}$ $\qquad \deg A = k \deg \mathcal{F}_\Gamma - n$

1. Compute a basis of the space S_k of all syzygies of degree $k \deg \mathcal{F}_\Gamma - n + 1$ quotiented by the space of trivial syzygies

$$D_{ij} = -D_{ji}, \qquad B_i = \sum_{j=1}^{n} D_{ij} \frac{\partial \mathcal{F}_\Gamma}{\partial x_j} \Longrightarrow \sum_i B_i \frac{\partial \mathcal{F}_\Gamma}{\partial x_i} = 0$$

are irrelevant because already used by the Griffiths–Dwork reduction

Extended Griffiths-Dwork : syzygies

Given a form $\Omega = \frac{A}{\mathcal{F}_\Gamma^k} d\underline{x}$ $\quad \deg A = k \deg \mathcal{F}_\Gamma - n$

2. Compute a normal form R of A modulo the Jacobian ideal plus the space $dV = \left\{ \sum_i \frac{\partial B_i}{\partial x_i} \,\middle|\, \underline{B} \in V \right\}$, that is for some polynomials B_i and C_i

$$A = R + \underbrace{\sum_i \frac{\partial B_i}{\partial x_i}}_{\in dV} + \underbrace{C_1 \frac{\partial \mathcal{F}_\Gamma}{\partial x_1} + \cdots + C_n \frac{\partial \mathcal{F}_\Gamma}{\partial x_n}}_{\in \text{Jacobian ideal}},$$

Extended Griffiths-Dwork : syzygies

Given a form $\Omega = \frac{A}{\mathcal{F}_\Gamma^k} d\underline{x}$ $\qquad \deg A = k \deg \mathcal{F}_\Gamma - n$

3. This leads to the following relation

$$(k-1)\frac{A}{\mathcal{F}_\Gamma^k} d\underline{x} = \frac{\sum_i \frac{\partial C_i}{\partial x_i}}{\mathcal{F}_\Gamma^{k-1}} d\underline{x} - d\left(\sum_i \frac{B_i}{\mathcal{F}_\Gamma^k}\xi_i + \sum_i \frac{C_i}{\mathcal{F}_\Gamma^{k-1}}\xi_i\right).$$

Then

$$\int_\gamma \frac{A(\underline{x})}{\mathcal{F}_\Gamma(\underline{x})^k} d\underline{x} = -\frac{1}{k-1}\int_\gamma \frac{\sum_i \frac{\partial C_i}{\partial x_i}}{\mathcal{F}_\Gamma^{k-1}} d\underline{x},$$

The extended Griffiths–Dwork reduction presented above is not always enough and may need further extensions, i.e. syzygies of syzygies. There is a hierarchy of extensions which eventually collapse to the strongest possible reduction.
However, for all the computations presented here, we only needed the first extension.

Pole conditions [Picard (1899)]

In the construction we will only consider the case where $\beta(\underline{x}, t)$ is holomorphic on $\mathbb{P}^{n-1} \setminus X_\Gamma$, that is is $\beta(\underline{x}, t)$ does not have poles that are not present in $\Omega(t; D, \underline{x})$.

Consider the rational function $F(x_1, x_2)$

$$\frac{ax_1 + bx_2 + c}{\left(\alpha x_1^2 + \beta x_2^2 + \gamma x_1 x_2 + \delta x_1 + \eta x_2 + \zeta\right)^2} = \partial_{x_1} \frac{N_1(x_1, x_2)}{D_1(x_1, x_2)} + \partial_{x_2} \frac{N_2(x_1, x_2)}{D_2(x_1, x_2)}$$

where $a, b, c, \alpha, \beta, \gamma, \delta, \eta, \eta$ are constants and polynomials $N_i(x_1, x_2)$ and $D_i(x_1, x_2)$ with $i = 1, 2$.

The denominators have poles at $x_2^0 = (a\delta - 2\alpha c)/(2\alpha b - a\gamma)$ which is not a pole of the left-hand-side.

This means one can find a cycle γ passing by x_2^0 such that the integral of $\int_\gamma F(x_1, x_2)$ is finite and non-vanishing.

Minimality of the Picard–Fuchs operator

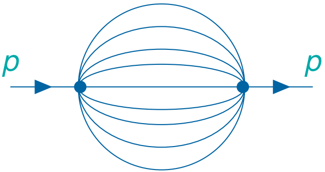
This dimension $\dim(V_\Gamma) = (-1)^{n+1}\chi\left((\mathbb{C}^*)^n \backslash \mathbb{V}(\mathcal{U}_\Gamma) \cup \mathbb{V}(\mathcal{F}_\Gamma)\right)$ gives an upper bound on the order of the minimal order differential operator acting on the Feynman integral.

The extended Griffiths–Dwork algorithm leads to a minimal order differential operator

$$\mathcal{L}_\Gamma \Omega_\Gamma = d\beta_\Gamma$$

annihilating (in cohomology) the Feynman integral differential form Ω_Γ with the condition that the certificate β_Γ is an holomorphic form on $\mathbb{P}^{n-1} \backslash Z_\Gamma$. $\mathcal{L}_\Gamma$ is the minimal differential order differential operator satisfying this condition.

Using the algorithm by [Chyzak, Goyer, Mezzarobba] we test the irreducibility of the the Picard–Fuchs operator and factorize when it is reducible.



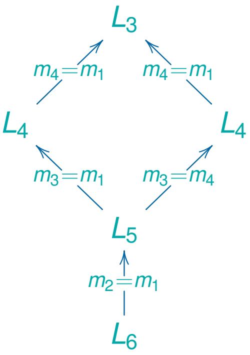
$$\Omega_n^{\ominus}(t,\underline{m}^2) := \frac{\Omega_0}{\mathcal{F}_n^{\ominus}(t,\underline{m}^2;\underline{x})} \in H^{n-1}(\mathbb{P}^{n-1}\backslash X_{\ominus})$$

$$\mathcal{F}_n^{\ominus}(t,\underline{m}^2;\underline{x}) := x_1\cdots x_n\left(\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)\left(\sum_{j=1}^{n} m_j^2 x_j\right)-t\right)$$

For generic physical parameters configurations we find a minimal order Picard–Fuchs operator

$$\mathcal{L}_t = \sum_{r=0}^{o_n} q_r(t,\underline{m}^2)\left(\frac{d}{dt}\right)^r \qquad o_n = 2^n - \binom{n+1}{\left\lfloor \frac{n+1}{2}\right\rfloor}; n \geqslant 2.$$

The three-loop sunset graph



$$L_r = (\alpha \frac{d}{dp^2} + \beta) \circ L_{r-1}$$

The Picard-Fuchs operators for the Feynman integral for general parameters $m_4 \neq m_1 \neq m_2 \neq m_3$

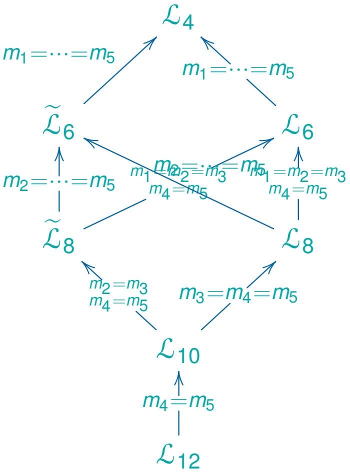
$$L_6 = \sum_{r=0}^{6} q_r(s) \left(\frac{d}{dp^2}\right)^r$$

is order 6 and degree 25

$$q_6(p^2) = \tilde{q}_6(p^2) \times \prod_{\epsilon_i = \pm 1} (p^2 - (\epsilon_1 m_1 + \epsilon_2 m_2 + \epsilon_3 m_3 + \epsilon_4 m_4)^2)$$

with $\tilde{q}_6(p^2)$ degree 17 with apparent singularities

The four-loop sunset graph



For generic kinematics we have an order 12 degree 121 operator

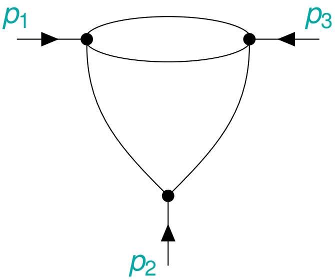
$$\mathcal{L}_t^{[1^5]} = \sum_{r=0}^{12} q_r^{[1^5]}(t, \underline{m}^2) \left(\frac{d}{dt}\right)^r .$$

When specialising masses we get the factorisation by an order 2 operator compatible with the change in the cohomology of the CY 3-fold

The five-loop and six-loop sunset graph

- **The six mass configuration** $m_1 \neq m_2 \neq m_3 \neq m_4 \neq m_5 \neq m_6$ **denote** $[1^6]$**:** the Picard–Fuchs operator of order 29 and degree of the polynomial $q_{29}(t)$ is 521.
- **The seven mass configuration** $m_1 \neq m_2 \neq m_3 \neq m_4 \neq m_5 \neq m_6 \neq m_7$**:** the Picard–Fuchs operator of order 58 with a degree 2273
- Results compatible with a CY $n-1$-fold

The ice-cream graph

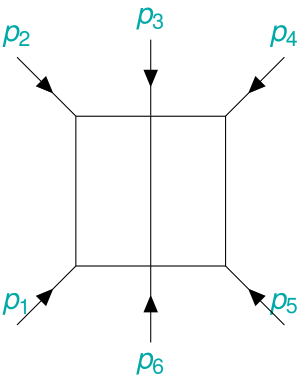


With a regular certificate the Picard–Fuchs operator is of order 2 and degree 9

$$\mathcal{L}_t^2 = q_0(t) + q_1(t)\frac{d}{dt} + q_2(t)\left(\frac{d}{dt}\right)^2,$$

$\mathcal{L}_t$ is a Liouvillian differential equations with only rational solutions

The double-box graph



We have the rational differential form in $\mathbb{P}^6$

$$\Omega(t) = \frac{\mathcal{U}_7(\underline{x})\Omega_0}{(\mathcal{U}_7(\underline{x})\mathcal{L}_7(\underline{m}^2, \underline{x}) - t\mathcal{V}_{\text{DoubleBox}}(\underline{s}, \underline{x}))^3}$$

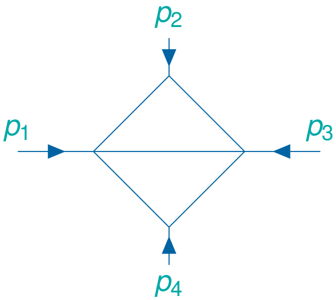
$$\mathcal{U}_7(\underline{x}) = (x_1 + x_2 + x_3)(x_4 + x_5 + x_6) + (x_1 + \cdots + x_6)x_7$$
$$\mathcal{L}_7(\underline{m}^2, \underline{x}) = m_1^2 x_1 + \cdots + m_7^2 x_7$$

$$\mathcal{V}_{\text{DoubleBox}}(\underline{s}, \underline{x}) = p_1^2 x_1 x_2 (x_4 + x_5 + x_6 + x_7) + \cdots$$

With $p_1, \ldots, p_6 \in \mathbb{C}^4$ we find a 2 order differential operator with monodromies (i.e. elliptic solutions)

Otherwise for $p_1, \ldots, p_6 \in \mathbb{C}^{D>4}$ the Picard–Fuchs operator of order 4 irreducible (i.e. hyperelliptic)

The kite graph



The rational differential form in $\mathbb{P}^4$

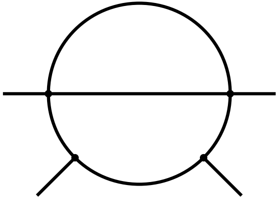
$$\Omega(t) = \frac{\Omega_0^{(5)}}{\mathcal{U}_5(\underline{x})(\mathcal{U}_5(\underline{x})\mathcal{L}_5(\underline{m}^2, \underline{x}) - t\mathcal{V}_{\rm kite}(\underline{s}, \underline{x}))}$$

The algorithm gives seven order operators that factorise

$$\mathcal{L}_{\rm kite} = L^{(1)} L^{(2)} L^{(3)} \left(z\frac{d}{dz} + 1 \right)$$

where $L^{(r)}$ with $r = 1, 2, 3$ are Liouvillian order 2 operators.

The observatory graph



The rational differential form in $\mathbb{P}^4$

$$\Omega(t) = \frac{\Omega_0^{(5)}}{\mathcal{U}_5(\underline{x})(\mathcal{U}_5(\underline{x})\mathcal{L}_5(\underline{m}^2,\underline{x}) - t\mathcal{V}_{\rm obser}(\underline{s},\underline{x}))}$$

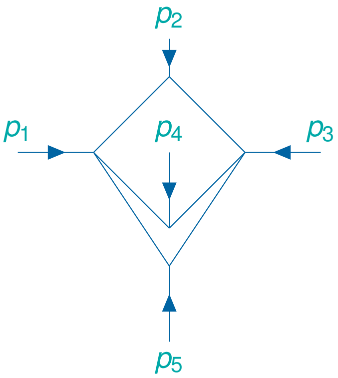
The algorithm gives 4th order operators that factorise

$$\mathcal{L}_{\rm obser} = L_2 L_1^a L_1^b$$

where L_2 is a second order operator with elliptic solutions and L_1^r with $r = a, b$ are first order operators.

Comparing with the kite we see how the structure of the graph polynomial (and its singularities) affects the PF operator

Tardigrade



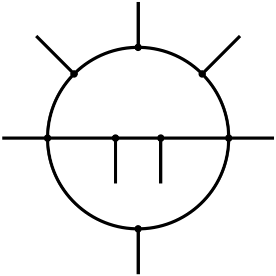
The rational differential form in $\mathbb{P}^5$

$$\Omega(t) = \frac{\Omega_0^{(6)}}{\left(\mathcal{U}_6(\underline{x})\mathcal{L}_6(\underline{m}^2, \underline{x}) - t\mathcal{V}(\underline{s}, \underline{x})\right)^2}$$

$$\mathcal{U}_6(\underline{x}) = (x_1 + x_2)(x_3 + x_4) + (x_1 + x_2)(x_5 + x_6) + (x_3 + x_4)(x_5 + x_6)$$

$V(\underline{s}, \underline{x}) = \sum_{1\leqslant i,j,k\leqslant 6} C_{ijk} y_i y_j y_k$ with linear changes $(x_{2i-1}, x_{2i}) \to (y_{2i-1}, y_{2i})$ and $i = 1, 2, 3$ C_{ijk} symmetric traceless i.e. $C_{iij} = 0$

The algorithm gives an irreducible Picard–Fuchs operator of order 11 with an head polynomial of degree up to 215. Compatible with K3 of Pic 11



For two-loop Feynman graphs of (a, b, c) vertices we consider the differential form

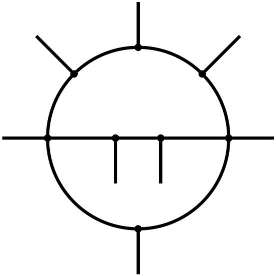
$$\Omega_{(a,b,c);D} = \frac{\mathbf{U}_{(a,b,c)}^{a+b+c-\frac{3D}{2}}}{\mathbf{F}_{(a,b,c);D}^{a+b+c-D}}\Omega_0$$

where $\deg \mathbf{U}_{(a,b,c)} = 2$ and $\deg \mathbf{F}_{(a,b,c);D} = 3$
The mixed Hodge structure

$$\mathrm{H}^{a+b+c-1}(\mathbb{P}^{a+b+c-1} - {}^{\mathbf{b}}Z_{(a,b,c);D}; B - (B \cap {}^{\mathbf{b}}Z_{(a,b,c);D}))$$

where $Z_{\Gamma;D}$ is the singular locus of the differential form

Motives for two-loop graphs



For two-loop Feynman graphs of (a, b, c) vertices we consider the differential form

$$\Omega_{(a,b,c);D} = \frac{\mathbf{U}_{(a,b,c)}^{a+b+c-\frac{3D}{2}}}{\mathbf{F}_{(a,b,c);D}^{a+b+c-D}}\Omega_0$$

The mixed Hodge structure

$$\mathrm{H}^{a+b+c-1}(\mathbb{P}^{a+b+c-1}\backslash {}^{\mathbf{b}}Z_{(a,b,c);D}; B\backslash(B \cap {}^{\mathbf{b}}Z_{(a,b,c);D}))$$

The method is based on quadric fibrations. The cohomology of $Z_{(a,b,c);D}$ is obtained by iterated extensions with Tate twists of cohomology of hyperelliptic curves and Tate Hodge structure

Definitions

At two-loop we encounter the following motives

Definition

1. Let $\mathbf{MHS}_{\mathbb{Q}}$ denote the abelian category of $\mathbb{Q}$-mixed Hodge structures.
2. The largest extension-closed subcategory of $\mathbf{MHS}_{\mathbb{Q}}$ containing the Tate twists of $\mathrm{H}^1(C;\mathbb{Q})$ for every hyperelliptic curve C is called $\mathbf{MHS}_{\mathbb{Q}}^{\mathrm{hyp}}$.
3. The largest extension-closed subcategory of $\mathbf{MHS}_{\mathbb{Q}}$ containing the Tate twists of $\mathrm{H}^1(E;\mathbb{Q})$ for every elliptic curve E is called $\mathbf{MHS}_{\mathbb{Q}}^{\mathrm{ell}}$.

Graph $(a, 1, c)$ I

$$\Omega_{(a,1,c);D} = \frac{\mathbf{U}_{(a,b,c)}^{a+1+c-\frac{3D}{2}}}{\mathbf{F}_{(a,1,c);D}^{a+1+c-D}}\Omega_0$$

Theorem

If $3D/2 \leqslant a + c$ *then*

$$\mathrm{H}^{a+c}(\mathbb{P}_\Gamma - {}^{\mathbf{b}}X_{(a,1,c);D}; B_\Gamma - (B_\Gamma \cap {}^{\mathbf{b}}X_{(a,1,c);D})) \in \mathbf{MHS}_{\mathbb{Q}}^{\mathrm{hyp}}.$$

In particular, this means that the Feynman integrals in all of these cases can be constructed from algebraic functions and periods of hyperelliptic curves.

Graph $(a, 1, c)$ II

Theorem

Suppose $a \leqslant 2$ or $c \leqslant 2$. Then

$$\mathrm{H}^{a+c-1}(X_{(a,1,c);D};\mathbb{Q}) \in \mathbf{MHS}_{\mathbb{Q}}^{\mathrm{ell}}.$$

Corollary

If $3D/2 \leqslant a+c$ and either $a \leqslant 2$ or $c \leqslant 2$ then

$$\mathrm{H}^{a+c}(\mathbb{P}_\Gamma - {}^{\mathbf{b}}X_{(a,1,c);D}; B - (B_\Gamma \cap {}^{\mathbf{b}}X_{(a,1,c);D})) \in \mathbf{MHS}_{\mathbb{Q}}^{\mathrm{ell}}.$$

Graph $(a, 1, c)$ III

This means that the mixed Hodge structure $\mathrm{H}^{a+c}(\mathbb{P}_\Gamma - {}^{\mathbf{b}}X_{(a,1,c);D}; B - (B \cap {}^{\mathbf{b}}X_{(a,1,c);D}))$ is constructed by taking iterated extensions of $\mathrm{H}^1(E;\mathbb{Q})(-a)^{r_1}$ and $\mathbb{Q}(-b)^{r_2}$ for different values of $a, b, r_1,$ and r_2, and with various possibly different elliptic curves. Therefore the Feynman integrals in this cases are built from algebraic and elliptic functions.

Graph $(3,1,3)$ double-box

$$\Omega_{(3,1,3);D}(t) = \frac{\mathcal{U}_{(3,1,3)}(\underline{x})\Omega_0}{(\mathcal{U}_{(3,1,3)}(\underline{x})\mathcal{L}_{(3,1,3)}(\underline{m}^2,\underline{x}) - t\mathcal{V}_{(3,1,3);D}(\underline{s},\underline{x}))^3}$$

Theorem

For arbitrary kinematic parameters, and arbitrary space-time dimension D, $W_4\mathrm{H}^5(X_{(3,1,3);D};\mathbb{Q})$ is mixed Tate.

1. *If $D \geqslant 6$ then $\mathrm{Gr}_5^W\mathrm{H}^5(X_{(3,1,3);D};\mathbb{Q}) \cong \mathrm{H}^1(C;\mathbb{Q})(-2)$ for a curve C which has genus 2 for generic kinematic parameters.*
2. *If $D = 4$ then $\mathrm{Gr}_5^W\mathrm{H}^5(X_{(3,1,3);D};\mathbb{Q}) \cong \mathrm{H}^1(E;\mathbb{Q})(-2)$ for a curve E which is elliptic for generic kinematic parameters.*
3. *If $D \leqslant 4$ then $\mathrm{H}^5(X_{(3,1,3);E};\mathbb{Q})$ is mixed Tate.*

In $D = 4$ the PF operator obtained by the extended Griffith–Dwork construction is *identical* to the one associated with the canonical differential form on the elliptic curve defined from the graph polynomial

Graph $(2,1,2)$ kite

$$\Omega_{(2,1,2);D}(t) = \frac{\Omega_0}{\mathcal{U}_{(2,1,2)}(\underline{x})(\mathcal{U}_{(2,1,2)}(\underline{x})\mathcal{L}_{(2,1,2)}(\underline{m}^2,\underline{x}) - t\mathcal{V}_{(2,1,2);D}(\underline{s},\underline{x}))}$$

Corollary

The Hodge structure on $\mathrm{Gr}_i^W \mathrm{H}^4(\mathbb{P}^4 - (X_{(2,1,2);4} \cup Y_{(2,1,2)}); \mathbb{Q})$ *is pure Tate except when* $i = 5$*, in which case,*

$$\mathrm{Gr}_5^W \mathrm{H}^4(\mathbb{P}^4 - (X_{(2,1,2);4} \cup Y_{(2,1,2)}); \mathbb{Q}) \cong \mathrm{H}^1(E; \mathbb{Q})(-2)$$

for an elliptic curve E *depending on the kinematic variables.*

The Picard–Fuchs operator $\mathcal{L}_{(2,1,2);4}$ constructed with the extended Griffiths-Dwork algorithm has singularities at the location of the vanishing of the discriminant of the elliptic curve E determined by the kite graph polynomial. But none of the operators are elliptic suggesting a split in the Hodge structure

Graph $(1,1,2)$ ice-cream

$$\Omega_{(1,1,2);D}(t) = \frac{\mathcal{U}_{(1,1,2)}(\underline{x})\Omega_0}{(\mathcal{U}_{(1,1,2)}(\underline{x})\mathcal{L}_{(1,1,2)}(\underline{m}^2,\underline{x}) - t\mathcal{V}_{(1,1,2);D}(\underline{s},\underline{x}))^2}$$

Proposition

- $\mathrm{H}^2(X_{(2,1,1);2}(t);\mathbb{Q})$ *is generically pure Tate.*
- *Let* $Z = V(Disc_{(2,1,1);D}(x,z,t)) \subseteq \mathbb{P}^1 \times \mathbb{A}^1$ *and we may define* $\mathbb{V} := \pi_* \underline{\mathbb{Q}}_Z$. *The local system* $\mathbb{V}$ *is isomorphic to the direct sum* $\mathbb{U}_1 \oplus \mathbb{U}_2 \oplus \underline{\mathbb{Q}}_B^2$ *where* $\mathbb{U}_1$ *is a rank 1 local system and* $\mathbb{U}_2$ *is a rank 2 local system*
- *For generic kinematic and mass parameters,* $Sol(\mathcal{L}_{(2,1,1);2})$ *is isomorphic to* $\mathbb{U}_2 \cong \mathbb{U}_2^\vee$.

Graph $(1,1,2)$ ice-cream : Picard–Fuchs operator

Lemma

Let $\mathbb{L}_1, \mathbb{L}_2$ *be local systems of rank 1, and suppose that* s_1 *and* s_2 *are sections of* $\mathbb{L}_1 \otimes \mathcal{O}_M$ *and* $\mathbb{L}_2 \otimes \mathcal{O}_M$ *respectively. If*

$$\mathcal{L}_{s_1} = \frac{d}{ds} - f_1(s), \qquad \mathcal{L}_{s_2} = \frac{d}{ds} - f_2(s)$$

are the differential equations associated to s_1 *and* s_2 *respectively, then the differential equation associated to the section* $s_1 \oplus s_2$ *of* $(\mathbb{L}_1 \oplus \mathbb{L}_2) \otimes \mathcal{O}_M$ *is*

$$\mathcal{L}_{s_1 \oplus s_2} = (f_1(s) - f_2(s))\frac{d^2}{ds^2} + (f_2(s)^2 - f_1(s)^2 - f_1'(s) + f_2'(s))\frac{d}{ds} + f_2(s)f_1'(s) - f_1(s)f_2'(s) + f_1(s)^2 f_2(s) - f_1(s)f_2(s)^2$$

Graph $(1,1,2)$ ice-cream : Picard–Fuchs operator

Proposition

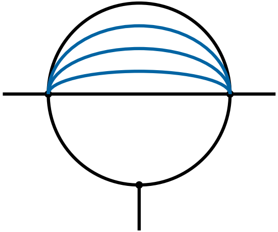
After the base change

$$t=\frac{(m_1-m_2)^2s^2+(m_1+m_2)^2}{p_2^2(s^2+1)},$$

the differential operator $\mathcal{L}_{(2,1,1);2}$ is of the form given previously where $f_i(s)$ with $i=1,2$ are obtained from the application of the change of variables $T=\rho_i(s)$ with $i=1,2$ to the differential operator

$$\frac{d}{dT}-\frac{(m_3^2+m_4^2-T)}{((m_3-m_4)^2-T)((m_3+m_4)^2-T)}\Longrightarrow\frac{d}{ds}-f_i(s)$$

with $T=\rho_i(s)$ and $i=1,2$ are roots of discriminant obtained from blowing up the linear subspace $x_1=z=0$



For the multiscoop ice cream cone families there is a conic fibration on $X_{(2,[1]^k);D}$, and the discriminant locus this conic fibration is a union of two Calabi–Yau $(k-2)$-folds associated to the $(k-1)$-loop sunset graph

Therefore $\mathrm{Gr}_k^W \mathrm{H}^k(X_{(2,[1]^k);2};\mathbb{Q})$ arises from

$$\mathrm{H}^{k-2}(X^{(1)}_{([1]^k);2};\mathbb{Q}) \oplus \mathrm{H}^{k-2}(X^{(2)}_{([1]^k);2};\mathbb{Q})$$

where $X^{(1)}_{([1]^k);2}$ and $X^{(2)}_{([1]^k);2}$ are distinct $(k-1)$-loop sunset Calabi–Yau $(k-2)$-folds.

This is supported by the computations of the PF operator for the 2-scoop ice-cream which is of rank 4. In this case $X^{(1)}_{(1,1,1);2}$ and $X^{(2)}_{(1,1,1);2}$ are elliptic curves, so the rank of $\mathcal{L}_{(2,1,1,1);2}$ agrees with the rank of $\mathrm{H}^1(X^{(1)}_{(1,1,1);2};\mathbb{Q}) \oplus \mathrm{H}^1(X^{(2)}_{(1,1,1);2};\mathbb{Q})$. This has been noticed for the all equal mass case by [Klemm, Duhr et al.]