RIGID IRREDUCIBLE MEROMORPHIC CONNECTIONS IN DIMENSION ONE

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by

Claude Sabbah

Abstract. I will illustrate the Arinkin-Deligne-Katz algorithm for rigid irreducible meromorphic bundles with connection on the projective line by giving motivicity consequences similar to those given by Katz for rigid local systems.

1. Rigidity and cohom. rigidity

Let $U \subsetneq \mathbb{P}^1$ be a Zariski open subset of the projective line, that we fix in this talk. Let (V, ∇) be an algebraic vector bundle with connection on U, possibly with irregular singularities at $\mathbb{P}^1 \smallsetminus U$. In this talk, I always assume that such bundles are *irreducible.* This property is preserved by the various transformations that will be applied.

At some point, one has to make a choice for $0, \infty \in \mathbb{P}^1$, and consider $\mathbb{G}_{\mathrm{m}} \subset \mathbb{A}^1 \subset \mathbb{P}^1$. For additive convolution and Fourier transformation, one has to work on $\mathbb{A}^1$. It is therefore useful to refer to the equivalence of categories

$$(V, \nabla) \text{ irred. bdle with conn. on } U \longleftrightarrow M \text{ irred. holonomic } \mathscr{D}_{\mathbb{A}^1}\text{-module}$$

$$(V, \nabla) \xmapsto{\text{min. extension}} M = j_{!*}\big[(V, \nabla)_{|U\cap\mathbb{A}^1}\big]$$

$$\text{with } j : U \cap \mathbb{A}^1 \hookrightarrow \mathbb{A}^1$$

$$(V, \nabla) = j'_{!*} M_{|U\cap\mathbb{A}^1} \longleftarrow\!\shortmid M \quad \text{with } j' : U \cap \mathbb{A}^1 \hookrightarrow U.$$

Example. P: diff. operator with polynomial coeffs. which is irreducible. Then $M = \mathbb{C}[z]\langle\partial_z\rangle/(P)$ is an irreducible holonomic $\mathbb{C}[z]\langle\partial_z\rangle$-module.

***Definition* (Katz).**

- (V, ∇) is *(physically) rigid* if for any (V', ∇') on U,

$$(V', \nabla')_{\widehat{x}} \simeq (V, \nabla)_{\widehat{x}}\ \forall\, x \in \mathbb{P}^1 \smallsetminus U \implies (V', \nabla') \simeq (V, \nabla).$$

- (V, ∇) is *cohom. rigid* if $\mathrm{rig}(V, \nabla) := \chi_{\mathrm{dR}}\big(\mathbb{P}^1, k_{!*}(\mathrm{End}(V, \nabla))\big) = 2$, i.e.,

$$h^1_{\mathrm{dR}}\big(\mathbb{P}^1, k_{!*}(\mathrm{End}(V, \nabla))\big) = 0, \quad k : U \hookrightarrow \mathbb{P}^1.$$

Remark. Katz (reg.) and Bloch-Esnault (irreg.) show that, on $\mathbb{P}^1$, cohom. rigid $\iff$ rigid. In higher dim., cohom. rigid $\implies$ rigid, but the converse is false.

***Example* (Regular singularity).** If (V,∇) has reg. sing. at each $x\in\mathbb{P}^1\smallsetminus U$, with loc. syst. V^∇ on U^{an}, then

- giving $(V,\nabla)_{\widehat{x}}$ is equivalent to giving the local monodromy of V^∇ at x,
- $h^1_{\mathrm{dR}}\big(\mathbb{P}^1, k_{!*}(\mathrm{End}(V,\nabla))\big)=h^1\big(\mathbb{P}^1,k_*(\mathrm{End}\,V^\nabla)\big)$,

and the notions of rigidity are those in the book of Katz.

Remark. In the *regular* case, rigidity concerns local monodromy data while in the *irregular* case, it only concerns *formal* monodromy data and not Stokes data. *Formal mon. data*: At x with loc. coord. z, $\exists\, p\geqslant 1$ s.t. after ramif. $\rho: t\mapsto z=t^p$, the matrix of ∇ on $\rho^*V_{\widehat{x}}$ has a Jordan normal form with blocks

$$(\mathrm{d}\varphi+\alpha\mathrm{d}t/t)\cdot\mathrm{Id}+N_\alpha\mathrm{d}t/t,\qquad \begin{cases}\varphi=\mathrm{exp.\,factor}\in\mathbb{C}(\!(t)\!)/\mathbb{C}[\![t]\!],\\ \alpha\in\mathbb{C},\ \ N_\alpha:\mathrm{cst.\,nilp.\,matrix}\\ \exp\Big(-2\pi\mathrm{i}\bigoplus_\alpha(\alpha\,\mathrm{Id}+N_\alpha)\Big)=\text{formal monodr.}\end{cases}$$

We say that (V,∇) is *quasi-unipotent* if each α is a rational number (i.e., $\exp(-2\pi\mathrm{i}\alpha)\in\mu_N$ for some N).

2. The Arinkin-Deligne-Katz algorithm

Katz proved in his book that (cohom.) rigidity of a regular (V,∇) is characterized by the existence of an algorithm composed of suitable tensor product with rank-one bundles with regular connection and middle convolutions with Kummer sheaves, starting from (V,∇) and abutting to a trivial bundle $(\mathcal{O}_{\mathbb{P}^1},\mathrm{d})$. He also suggests that a similar algorithm should exist in the irregular case, with Fourier transformation replacing middle convolution. Main property for proving that:

***Theorem* (Katz, Bloch-Esnault).** *For (V,∇) irreducible, the index of rigidity is preserved by middle convolution with a Kummer sheaf and by Fourier transformation.*

***Theorem* (Katz, Arinkin-Deligne).** *If (V,∇) is irreducible, it is rigid iff it can be reduced to $(\mathcal{O}_{\mathbb{P}^1},\mathrm{d})$ by a suitable sequence of transormations:*

(1) tensoring (V,∇) with a rank-one bundle with connection on U,

(2) pullback by a coordinate change on $\mathbb{P}^1$ (homography),

(3) Fourier transformation.

3. Three theorems

These theorems are motivated by there analogues for regular singularities, due to Katz and, in higher dimension, Esnault-Gröchenig.

***Theorem A* (Finiteness).**

- *Let Φ be a finite set of ramified polar parts:*

$$\varphi \in \Phi \iff \varphi(z) \in \mathbb{C}(\!(z^{1/p})\!)/\mathbb{C}[\![z^{1/p}]\!] \quad \text{for some } p \geqslant 1.$$

- *Let r, N be integers $\geqslant 1$.*

Then there exists a finite number of irreducible rigid (V, ∇) on U

- *of rank $\leqslant r$,*
- *which are quasi-unipotent of order dividing N at each $x \in \mathbb{P}^1 \smallsetminus U$ (i.e., formal monodromy has eigenvalues in μ_N),*
- *with exponential factors at each $x \in \mathbb{P}^1 \smallsetminus U$ contained in Φ.*

***Definition* (exponential-geometric origin).** We say that an algebraic vector bundle with an integrable connection (V, ∇) on $U \subsetneq \mathbb{P}^1$ is *of exponential-geometric origin* if, up to shrinking U, there exist a smooth morphism $f : Y \to U$ from a smooth quasi-projective variety and a regular function g on Y such that (V, ∇) is a subbundle of the "twisted Gauss-Manin connection" $\mathscr{H}^k f_+(\mathcal{O}_Y^r, \mathrm{d} + \mathrm{d}g)$ for some $r \geqslant 1$ and some $k \in \mathbb{Z}$.

Roughly speaking, horizontal sections (or solutions) of such a (V, ∇) on U^{an} can be given an integral expression, with the integrand being of the form $\mathrm{e}^g \cdot \omega$ for some algebraic differential form ω.

Theorem B. *Any quasi-unipotent rigid irreducible (V, ∇) on U has exponential-geometric origin.*

Irregular R-H correspondence. As formulated by Deligne-Malgrange:
$\varpi : \widetilde{\mathbb{P}}^1 \to \mathbb{P}^1$ real oriented blowing up of $\mathbb{P}^1$ at each $x \notin U$. $S^1_x := \varpi^{-1}(x)$. Any local system on U^{an} extends in a unique way as a local system on $\widetilde{\mathbb{P}}^1$.

$$(V, \nabla) \text{ of rank } r \iff \begin{cases} \mathbb{C}\text{-loc. syst. of rk } r \text{ on } U^{\mathrm{an}} \\ + \textit{Stokes filtr.} \text{ of the induced loc. syst. on each } S^1_x \end{cases}$$

Definition. A $\mathbb{C}$-Stokes-filtered local system is *integral* if it comes by extension of scalars from a $\overline{\mathbb{Z}}$-Stokes-filtered local system (i.e., monodromies and Stokes matrices are defined over $\overline{\mathbb{Z}}$).

***Theorem C* (Integrality).** *The Stokes-filtered local system associated to any quasi-unipotent, rigid, irreducible (V,∇) on U is integral.*

Example. Confluent hypergeometric equations (with non-resonant parameters) are irreducible and rigid. They are quasi-unipotent whenever the coefficients are rational numbers.
K. Jakob has classified rigid irreducible connections on $\mathbb{G}_{\mathrm{m}}$ whose slope at infinity takes the form $1/n$ ($n \in \mathbb{N}^*$) and with differential Galois group G_2. They depend on a set of parameters in $\mathbb{C}^*$. Such a connection is quasi-unipotent if the corresponding parameters are rational.

4. Ideas and sketch of the proofs

All the proofs rely on the extension by Arinkin and Deligne (independently) of the algorithm of Katz for rigids in the regular singular case. Because of that, they remain specific to $\mathbb{P}^1$. They use three distinct arguments.

***Idea of the proof of Theorem* A.** Analysis of the change of (U, N, r, Φ) in each step of the A-D-K algorithm. Uses the formal stationary phase formula and joint work with M. Dettweiler.

Geometric formulation of the A-D-K algorithm. One expresses the overall result of the A-D-K algorithm as a geometric transformation in higher dimensions. It is more convenient to work on $\mathbb{P}^1$ and set $\mathcal{M} = k_{!*}(V,\nabla)$. The idea is: for each step in the algorithm, yielding a transformation of an object on $\mathbb{P}^1$ to an object on $\mathbb{P}^1$, regard it as an integral transformation with some kernel

$$\begin{array}{ccc} & \mathbb{P}^1 \times \mathbb{P}^1 & \\ \swarrow & & \searrow \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{array}$$

Then there exist

(a) a smooth projective complex variety X and a strict normal crossing divisor $D \subset X$, together with a subdivisor $D_1 \subset D$,

(b) a projective morphism $f : X \to \mathbb{P}^1$,

(c) a rational function g on X with poles contained in D and whose pole and zero divisors do not intersect,

(d) a locally free rank-one $\mathcal{O}_X(*D)$-module $\mathcal{N} = \mathcal{N}_{\mathrm{reg}}$ with a regular singular meromorphic connection ∇, such that $\mathcal{N}$ is of torsion (i.e., $\mathcal{N}^{\otimes N} \simeq (\mathcal{O}_X(*D), \mathrm{d})$ for some $N \geqslant 1$) and $\mathcal{M}$ is the image of the natural morphism

$$\mathcal{H}^0 f_+\big[(\mathcal{N}, \nabla + \mathrm{d}g)(!D_1)\big] \longrightarrow \mathcal{H}^0 f_+(\mathcal{N}, \nabla + \mathrm{d}g).$$

***Sketch of the proof of Theorem* B.** Enough to show that $\mathcal{H}^0 f_+(\mathcal{N}, \nabla + \mathrm{d}g)$ has exponential-geometric origin. This would be clear if $\mathcal{N} = \mathcal{O}_X(*D)$. However, we only know that $\mathcal{N}^{\otimes N} \simeq \mathcal{O}_X(*D)$. An argument of Kawamata yields a proper modification $\pi : X' \to X$ which is finite étale over $X \smallsetminus D$ such that $\pi^*\mathcal{N} \simeq \mathcal{O}_{X'}(*D')$ ($D' = \pi^{-1}(D)$ is also a sncd).
Set $f' = f \circ \pi : X' \to \mathbb{P}^1$. Then $\mathcal{H}^0 f'_+(\mathcal{O}_{X'}(*D'), \nabla + \mathrm{d}(g \circ \pi))$ is of exponential-geometric origin and one uses that $\mathcal{N}$ is a direct summand in $\pi_*\mathcal{O}_{X'}(*D')$.

***Sketch of the proof of Theorem* C.** In this case, one only needs to show that each term $\mathcal{H}^0 f_+\big[(\mathcal{N}, \nabla + \mathrm{d}g)(!D_1)\big]$ and $\mathcal{H}^0 f_+(\mathcal{N}, \nabla + \mathrm{d}g)$ is integral. Let us check the second one for example.

- First, integrality of the local system: $U \subset \mathbb{P}^1$ s.t. $f : (X_U, D_U) \to U$ is smooth. $\varpi : \widetilde{X}_U \to U^{\mathrm{an}}$: real oriented blow-up of X_U along the components of D_U, and $\widetilde{f}_U = f \circ \pi : \widetilde{X}_U \to U^{\mathrm{an}}$.
  $(V, \nabla) := \mathcal{H}^0 f_{U+}(\mathcal{N}, \nabla + \mathrm{d}g)_{|U}$.
  $\alpha : U^{\mathrm{an}} \hookrightarrow \widetilde{X}_U^{\mathrm{mod}}$: open subset of points near which e^{-g} has moderate growth.
  Then $(*)$ $\quad V^\nabla \simeq R^{n-1}\widetilde{f}_!(\alpha_*\mathcal{N}^\nabla)$, $\quad \widetilde{f} : \widetilde{X}_U^{\mathrm{mod}} \to U^{\mathrm{an}}$.
  $\implies L := V^\nabla$ defined over $\mathbb{Z}[\zeta]$.
- Second: extend by considering the real oriented blow up of X along all the components of D. $\rightsquigarrow \widetilde{f} : \widetilde{X} \to \widetilde{\mathbb{P}}^1$.
  For $x \in \mathbb{P}^1 \smallsetminus U$, $S^1_x \subset \widetilde{\mathbb{P}}^1$ (directions around x). $L_{|S^1_x}$ has a filtration by subsheaves: the Stokes filtration (Deligne-Malgrange) indexed by some $\Phi \subset \mathbb{C}(\!(z^{1/p})\!)/\mathbb{C}[\![z^{1/p}]\!]$. For each $\varphi \in \Phi$, $L_{\leqslant\varphi}$ is a subsheaf of $L_{|S^1_x}$.
  A theorem of Mochizuki implies $L_{\leqslant 0}$ defined by a formula similar to that of $(*)$.
  $\rightsquigarrow$ integrality of $L_{\leqslant 0}$.
  Then a similar argument for $L_{\leqslant\varphi}$ by replacing g with $g + \varphi \circ f$.