Sheaves for causal manifolds

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1 Introduction

Microlocal Analysis was created by Mikio Sato in the 70s, giving rise to the fundamental paper [SKK73], then to microlocal sheaf theory, introduced in [KS82] and developed in [KS90].

With Benoit Jubin [JS16] (see also [Sch26]), we have shown that this latter theory opens up new perspectives for the study of spacetimes. A spacetime is a connected manifold M endowed with a non-degenerate quadratic form of signature $(+,-,\dots,-)$. This form defines a cone in the tangent space TM, the affine model being the Minkowski space $\mathbb{R}^4$. Classical questions are:
(i) to define the future of $x \in M$ (the light cone issued from x). A natural answer being to consider piecewise smooth paths issued from x whose derivative belong to the cone,
(ii) to define and solve the Cauchy problem for various wave equations associated to the quadratic form.
There is an enormous literature on this subject (see some items in the bibliography).

Here, we do not use the quadratic form and start with a closed convex proper cone in the cotangent bundle $\lambda \subset T^*M$. We call (M, λ) a causal manifold and treat the above questions by using microlocal sheaf theory. We introduce the λ-topology, define the future of a set as its closure for this topology and define a time-function as a causal map $q\colon M \to \mathbb{R}$ proper on the future or the past of any point. Our main result is that when (M, λ) is endowed with a time-function (which corresponds to the classical notion of a globally hyperbolic spacetime), then the Cauchy problem is globally well-posed for hyperbolic systems in the framework of Sato's hyperfunctions.

2 Causal manifolds

To a real manifold M are associated its tangent bundle $\tau\colon TM \to M$ and its cotangent bundle $\pi\colon T^*M \to M$. For a submanifold $N \subset M$, one denotes by T_NM its normal bundle and by T^*_NM its conormal bundle. In particular, T^*_MM is the zero-section of T^*M.

Recall that to a morphism of manifolds $f\colon M \to N$, one associates the maps

(2.1)
$$\begin{array}{ccccc} TM & \xrightarrow{f'} & M \times_N TN & \xrightarrow{f_\tau} & TN \\ & \searrow^{\tau} & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ & & M & \xrightarrow{f} & N, \end{array} \qquad \begin{array}{ccccc} T^*M & \xleftarrow{f_d} & M \times_N T^*N & \xrightarrow{f_\pi} & T^*N \\ & \searrow^{\pi} & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ & & M & \xrightarrow{f} & N. \end{array}$$

We set

(2.2)
$$Tf := f_\tau \circ f' \colon TM \to TN,$$

and call Tf the tangent map to f.

For a cone γ in a vector bundle $E \to M$, one denotes by γ^a the cone $-\gamma$ and by γ° the polar cone, a closed convex cone of E^*.

Consider a closed cone $\lambda \subset T^*M$ stisfying:

(2.3) λ is a closed convex proper cone of T^*M and $T^*_MM \subset \lambda$.

Note that

(2.4)
$$\lambda^a \cap \lambda = T^*_MM, \quad \lambda + \lambda = \lambda.$$

Definition 2.1. (a) A causal manifold (M, λ) is a manifold M equipped with a cone $\lambda \subset T^*M$ satisfying (2.3).

(b) A morphism of causal manifolds $f\colon (M, \lambda_M) \to (N, \lambda_N)$ is a morphism of manifolds $f\colon M \to N$ satisfying $f_d f_\pi^{-1}\lambda_N \subset \lambda_M$. One also says that the morphism f is causal.

The next result gives a more intuitive version of the notion of a causal morphism.

Proposition 2.2 (see [JS16, Prop. 1.12])**.** *Given two causal manifolds (M, λ_M) and (N, λ_N), a morphism of manifolds $f\colon M \to N$ is causal if and only if $Tf\colon TM \to TN$ sends λ_M° into λ_N°, that is, $Tf(\lambda_M^\circ) \subset \lambda_N^\circ$.*

We can now define the notion of a causal path.

Let I be the interval $[0,1]$ of the real line $\mathbb{R}$, endowed with the coordinate t. Denoting by $(t;\tau)$ the associated coordinates on $T^*\mathbb{R}$, we denote for short by $(\mathbb{R},+)$ the causal manifold associated with the cone

$$\Lambda_{\mathbb{R}} = \{(t;\tau); \tau \geq 0\}.$$

Notation 2.3. If a function $c\colon I \to M$ is left (resp. right) differentiable, we denote its left (resp. right) differential by c'_l (resp. c'_r).

Definition 2.4. (a) A path $c\colon I \to M$ is a piecewise smooth map.

(b) If (M, λ_M) is a causal manifold, the path c is *causal* if $c'_l(t), c'_r(t) \in (\lambda^\circ_M)_{c(t)}$ for any $t \in I$.

(c) One denotes by $\underset{ps}{\preceq}$ the preorder given by $x \underset{ps}{\preceq} y$ if there exists a causal path $c\colon I \to M$ with $c(0) = x$ and $c(1) = y$ and calls it the piecewise smooth preorder, ps-preorder for short. One denotes by $J^+_{\mathrm{ps}}(A)$ and $J^-_{\mathrm{ps}}(A)$ the future and past sets of A for the ps-preorder.

Example 2.5. (i) Let $\mathbb{V}$ be a real finite dimensional vector space and let θ be a closed proper convex cone with non-empty interior. Let V be an open convex subset. Set $\gamma = V \times \theta \subset T\mathbb{V}$ and $\lambda = V \times \gamma^{\circ a}$. Then (V, γ) is a causal manifold
(ii) For I an open interval of $\mathbb{R}$, we denote by $(I,+)$ the causal manifold (I, λ) where $\lambda = \{(t;\tau) \in T^*I; \tau \geq 0\}$.

3 Microlocal sheaf theory (after [KS90])

Let $\mathbf{k}$ be a unital commutative ring with finite global dimension. For a topological space M, denote by $\mathrm{D^b}(\mathbf{k}_M)$ the bounded derived category of sheaves of $\mathbf{k}$-modules on a real manifold M.

Definition 3.1. Assume that M is a real $\mathscr{C}^\infty$-manifold and let $F \in \mathrm{D^b}(\mathbf{k}_M)$. The micro-support $\mathrm{SS}(F)$ of F is the closed $\mathbb{R}^+$-conic subset of T^*M defined as follows: for an open subset $W \subset T^*M$ one has $W \cap \mathrm{SS}(F) = \emptyset$ if and only if for any $x_0 \in X$ and any real $\mathscr{C}^1$-function φ on M defined in a neighborhood of x_0 with $(x_0; d\varphi(x_0)) \in W$, one has $(\mathrm{R}\Gamma_{\{x;\varphi(x)\geq\varphi(x_0)\}}F)_{x_0} \simeq 0$.

In other words, $p \notin \mathrm{SS}(F)$ if the sheaf F has no cohomology supported by "half-spaces" whose conormals are contained in a neighborhood of p. The micro-support may be viewed as the set of co-directions of "non-propagation" of F.

- By its construction, the micro-support is $\mathbb{R}^+$-conic, that is, invariant by the action of $\mathbb{R}^+$ on T^*M.
- $\mathrm{SS}(F) \cap T^*_M M = \pi(\mathrm{SS}(F)) = \mathrm{supp}(F)$.

- The micro-support satisfies the triangular inequality: if $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ is a distinguished triangle in $\mathrm{D^b}(\mathbf{k}_M)$, then $\mathrm{SS}(F_i) \subset \mathrm{SS}(F_j) \cup \mathrm{SS}(F_k)$ for all $i, j, k \in \{1, 2, 3\}$ with $j \neq k$.

- The micro-support $\mathrm{SS}(F)$ is **co-isotropic** (or "involutive") [KS90, Def. 6.5.1].

In the sequel, for a locally closed subset $A \subset M$, we denote by $\mathbf{k}_A$ the sheaf on M which is the constant sheaf with stalk $\mathbf{k}$ on A and is zero on $M \setminus A$.

Example 3.2. (i) If F is a non-zero local system on M and M is connected, then $\mathrm{SS}(F) = T^*_M M$.
(ii) If N is a closed submanifold of M and $F = \mathbf{k}_N$, then $\mathrm{SS}(F) = T^*_N M$, the conormal bundle to N in M.
(iii) When $M = \mathbb{R}$, $F = \mathbf{k}_I$, I an interval:

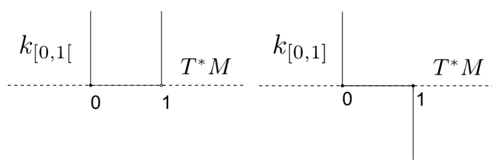

Figure 1: Examples

(iv) If X is a complex manifold and $\mathscr{M}$ a coherent $\mathscr{D}_X$-module, $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$, then $\mathrm{SS}(F) = \mathrm{char}(\mathscr{M})$, the characteristic variety of $\mathscr{M}$ (see § 6).

Operations (see [KS90])

Let $f\colon M \to N$ be a morphism of $\mathscr{C}^\infty$-manifolds.

Theorem 3.3. *Let $F \in \mathrm{D^b}(\mathbf{k}_M)$ and let $G \in \mathrm{D^b}(\mathbf{k}_N)$.*

(a) *Assume that f is proper on the support of F. Then $\mathrm{SS}(\mathrm{R}f_*F) \subset f_\pi f_d^{-1}\,\mathrm{SS}(F)$.*

(b) *Assume that f is non-characteristic with respect to G, that is, f_d is proper on $f_\pi^{-1}\,\mathrm{SS}(G)$. Then $\mathrm{SS}(f^{-1}G) \cup \mathrm{SS}(f^!G) \subset f_d f_\pi^{-1}\,\mathrm{SS}(G)$.*

There is also a result for the case where the map f is characteristic but the formulation is not so simple (and the proof is more difficult). We only formulate it for closed embeddings.

Consider a vector bundle $\tau\colon E \to M$. It gives rise to a morphism of vector bundles over M, $\tau'\colon TE \to E \times_M TM$ which by duality gives the map $\tau_d\colon E \times_M T^*M \to T^*E$. By restricting to the zero-section of E, we get the map:

$$T^*M \hookrightarrow T^*E.$$

Assume now that M is a closed submanifold of a real manifold X. Applying this construction to the bundle $T^*_MX \to M$, and using the Hamiltonian isomorphism we get the maps

$$T^*M \hookrightarrow T^*T^*_MX \simeq T_{T^*_MX}T^*X. \tag{3.1}$$

If (x,y) is a local coordinate system on X such that $M = \{y = 0\}$ and $(x,y;\xi,\eta)$ are the associated local coordinates on T^*X, then $T^*_MX = \{y = \xi = 0\}$. Denoting by $(x,\eta;\xi,y)$ the local coordinate system associated with the coordinates (x,η) on T^*_MX, the embedding (3.1) is described by:

$$(x;\xi) \mapsto (x,0;\xi,0).$$

To a subset A of T^*X, one associates $C_{T^*_MX}(A)$, the Withney normal cone of A along the submanifold T^*_MX. Hence $C_{T^*_MX}(A)$ is a closed subset of $T_{T^*_MX}T^*X$ identified to $T^*T^*_MX$ by (3.1).

Definition 3.4. Let $F \in \mathrm{D^b}(\mathbf{k}_X)$. One sets

$$\mathrm{SS}_M(F) = T^*M \cap C_{T^*_MX}(\mathrm{SS}(F)). \tag{3.2}$$

Denote by $j_M \colon M \hookrightarrow X$ the embedding.

Theorem 3.5 (see [KS90, Cor. 6.4.4 and Prop. 6.2.4])**.** *Let* $F \in \mathrm{D^b}(\mathbf{k}_X)$*. Then*

$$\mathrm{SS}(j_M^{-1}F) \cup \mathrm{SS}(j_M^!F) \subset \mathrm{SS}_M(F).$$

4 λ-topology

Definition 4.1. Let (M,λ) be a causal manifold.

(a) A locally closed set $A \subset M$ is a λ-set if $\mathrm{SS}(\mathbf{k}_A) \subset \lambda$.

(b) A λ-open set (resp. a λ-closed set) is an open set (resp. a closed set) which is also a λ-set.

Note that U is λ-open if and only if $M \setminus U$ is λ-closed.

Proposition 4.2. *Let* (M,λ) *be a causal manifold.*

(a) *The family of λ-open sets is closed under arbitrary unions and finite intersections. In particular, the family of λ-open subsets of M defines a topology on M.*

(b) *Similarly, the family of λ-closed sets is closed under arbitrary intersections and finite unions.*

Proof. Since an open set U is λ-open if and only if $M \setminus U$ is λ-closed, it is enough to prove (a).

(i) If U_1 and U_2 are λ-open, so is $U_1 \cap U_2$ since $\mathbf{k}_{U_1 \cap U_2} \simeq \mathbf{k}_{U_1} \otimes \mathbf{k}_{U_2}$ and $\mathrm{SS}(\mathbf{k}_{U_1} \otimes \mathbf{k}_{U_2}) \subset \lambda$ by (2.4).

(ii) If U_1 and U_2 are λ-open, so is $U_1 \cup U_2$. This follows from (i) and the distinguished triangle $\mathbf{k}_{U_1 \cap U_2} \to \mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2} \to \mathbf{k}_{U_1 \cup U_2} \xrightarrow{+1}$.

(iii) Let $\{U_i\}_{i \in I}$ be a family of λ-open subsets. Let us order I by the relation $i \le j$ if $U_i \subset U_j$. We may assume that I is non empty and by (ii) we may assume that $(I, \le)$ is directed. It then follows from [KS90, Exe. 5.7] that, setting $U = \bigcup_{i \in I} U_i$, $\mathrm{SS}(\mathbf{k}_U) \subset \lambda$.

(iv) Since $\mathrm{SS}(\mathbf{k}_M) = T^*_M M \subset \lambda$ and $\mathrm{SS}(\mathbf{k}_\varnothing) = \varnothing$, both M and $\varnothing$ are λ-open and the proof is complete. Q.E.D.

Example 4.3. Let $\mathbb{V}$, V and γ be as in Example 2.5. Set $\lambda = \gamma^{\circ a}$. Then an open set U of V is λ-open if and only if $U = (U + \theta) \cap V$.

Definition 4.4. For $A \subset M$, we denote by $L^+_\lambda(A)$ the closure of A for the λ-topology. In other words

$$L^+_\lambda(A) = \bigcap Z \text{ with } A \subset Z \text{ and } Z \text{ is } \lambda\text{-closed}.$$

For $x \in M$, we set $L^+_\lambda(x) = L^+_\lambda(\{x\})$.

Let us set

$$x \preceq_\lambda y \text{ if and only if } y \in L^+_\lambda(x).$$

Clearly, the relation $x \preceq_\lambda y$ is a preorder. It is the closure of the relation $\underset{ps}{\preceq}$.

5 Cauchy time functions on causal manifolds

The terminology G-causal below is not inspired by gravitation but by the name of Geroch.

Definition 5.1. (a) A *Cauchy time function* on a causal manifold (M, λ) is a submersive causal morphism $q\colon (M, \lambda) \to (\mathbb{R}, +)$ which is proper on the sets $L^+_\lambda(K)$ and $L^-_\lambda(K)$ for any compact set $K \subset M$.

(b) A *G-causal manifold* (M, γ, q) is the data of a causal manifold (M, γ) together with a Cauchy time function q.

One easily check that q is a time function as soon as it is proper on the sets $L^+_\lambda(x)$ and $L^-_\lambda(x)$ for all $x \in M$, in other words, diamonds are compact.

It follows from Definition 2.1 that

$$dq(t) \in \lambda \cup \lambda^a. \tag{5.1}$$

Definition 5.2. A Lorentzian spacetime (or a causal manifold) is globally hyperbolic if diamonds are compact and there are no causal loops.

Diamonds are the set of the form $J^+_{\mathrm{ps}}(A)$ and $J^-_{\mathrm{ps}}(B)$ with A and B compact.

Theorem 5.3. *If a Lorentzian spacetime is globally hyperbolic, then it admits a Cauchy time function.*

In the sequel, we denote by t a coordinate on $\mathbb{R}$ and by $(t;\tau)$ the associated coordinates on $T^*\mathbb{R}$. We shall write for short $\{\tau \geq 0\}$ instead of $\{(t;\tau) \in T^*\mathbb{R}; \tau \geq 0\}$ and similarly with $\tau \leq 0$.

Lemma 5.4. *Let (M, γ, q) be a G-causal manifold and let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$. Assume that $\mathrm{SS}(F) \cap \lambda \subset T^*_M M$. Then*

$$\mathrm{SS}(\mathrm{R}q_* F) \subset \{\tau \leq 0\}. \tag{5.2}$$

Sketch of proof. (i) If q is proper on $\mathrm{supp}\, F$, this result follows immediately from [KS90].
(ii) We choose an exhausting family of compact sets $\{K_n\}_n$, set $Z_n = L^+_\lambda(K_n)$ and apply(i) to F_{Z_n}, using the fact that $\mathrm{SS}(F_{Z_n}) \cap \lambda \subset T^*_M M$. Q.E.D.

Theorem 5.5. *Let (M, γ, q) be a G-causal manifold and let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$. Assume that $\mathrm{SS}(F) \cap (\lambda \cup \lambda^a) \subset T^*_M M$. Let $t_0 \in q(M)$ and set $M_0 = q^{-1}(t_0)$. Then M_0 is non-characteristic w.r.t. F, that is, $T^*_{M_0} M \cap \mathrm{SS}(F) \subset T^*MM$ and the natural restriction morphism below is an isomorphism:*

$$\mathrm{R}\Gamma(M; F) \xrightarrow{\sim} \mathrm{R}\Gamma(M_0; F|_{M_0}). \tag{5.3}$$

Sketch of proof. It follows from (5.2) that $\mathrm{SS}(\mathrm{R}q_* F) \subset T^*_{\mathbb{R}}\mathbb{R}$. Hence $\mathrm{R}q_* F$ is a constant sheaf on $\mathbb{R}$ and $\mathrm{R}\Gamma(M; F) \simeq \mathrm{R}\Gamma(\mathbb{R}; \mathrm{R}q_* F) \simeq \mathrm{R}q_* F|_{t=0} \simeq \mathrm{R}\Gamma(M_0; F|_{M_0})$. (The last isomorphism uses the fact that M_0 is non-characteristic for F.) Q.E.D.

6 $\mathscr{D}$-modules

For an exhaustive exposition, see [Kas03].

Consider first a finite system of linear equations over a (not necessarily commutative) ring $\mathscr{R}$:

$$\sum_{i=1}^{N_0} P_{ji} u_i = v_j, \quad j = 1, \ldots, N_1. \tag{6.1}$$

Here u_i and v_j belong to some left $\mathscr{R}$-module $\mathscr{S}$ and P_{ji} belongs to $\mathscr{R}$. Denote by P_0 the matrix $(P_{ji})_{1 \leq i \leq N_0, 1 \leq j \leq N_1}$ and by $P_0\cdot$ this matrix acting on the left on $\mathscr{S}^{N_0}$:

$$\mathscr{S}^{N_0} \xrightarrow{P_0\cdot} \mathscr{S}^{N_1}.$$

Now consider $\cdot P_0$, the matrix P_0 acting on the right on $\mathscr{R}^{N_0}$, and denote by $\mathscr{M}$ its cokernel, so that we have *an exact sequence*:

$$\mathscr{R}^{N_1} \xrightarrow{\cdot P_0} \mathscr{R}^{N_0} \to \mathscr{M} \to 0. \tag{6.2}$$

Conversely, consider a left $\mathscr{R}$-module $\mathscr{M}$ and assume that there exists an exact sequence (6.2). Then, one says that $\mathscr{M}$ admits a finite 1-presentation, but such a presentation is not unique and different matrices with entries in $\mathscr{R}$ may give isomorphic modules. This is similar to the fact that a finite dimensional vector space may have different systems of generators. As we shall see, when analyzing the system (6.1), the important (and "intrinsic") information is not the matrix P_0 but the module[1] $\mathscr{M}$.

Applying the contravariant left exact functor $\mathrm{Hom}_{\mathscr{R}}(\bullet, \mathscr{S})$ to (6.2) we find the exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{R}}(\mathscr{M}, \mathscr{S}) \to \mathscr{S}^{N_0} \xrightarrow{P_0 \cdot} \mathscr{S}^{N_1},$$

which shows that the kernel of $P_0\cdot$ depends only on $\mathscr{M}$ (up to isomorphism) not on the presentation (6.2).

Assume now that $\mathscr{R}$ is right Noetherian. Then the kernel of $\cdot P_0$ in (6.2) is finitely generated and one can extend the exact sequence (6.2) to a long exact sequence

$$\cdots \to \mathscr{R}^{N_2} \xrightarrow{\cdot P_1} \mathscr{R}^{N_1} \xrightarrow{\cdot P_0} \mathscr{R}^{N_0} \to \mathscr{M} \to 0. \tag{6.3}$$

Consider the complex

$$\mathscr{M}^\bullet := \cdots \to \mathscr{R}^{N_2} \xrightarrow{\cdot P_1} \mathscr{R}^{N_1} \xrightarrow{\cdot P_0} \mathscr{R}^{N_0} \to 0.$$

One proves that the complex $\mathrm{Hom}_{\mathscr{R}}(\mathscr{M}^\bullet, \mathscr{S})$, viewed as an object of the derived category, does not depend on the choice of the free resolution $\mathscr{M}^\bullet$ and one sets

$$\mathrm{RHom}_{\mathscr{R}}(\mathscr{M}, \mathscr{S}) = \mathrm{Hom}_{\mathscr{R}}(\mathscr{M}^\bullet, \mathscr{S}). \tag{6.4}$$

These constructions may naturally be generalized when $\mathscr{R}$ is no more a ring but is a sheaf of rings. In particular, if X is a complex manifold, there is the sheaf $\mathscr{D}_X$ of (finite order) holomorphic differential operators. In a local coordinate system $(x_1, \dots, x_n)$ a differential operator P of order $\leq m$ may be written as a polynomial

$$P = \sum_{|\alpha| \leq m} a_\alpha \partial_x^\alpha.$$

A coherent left $\mathscr{D}_X$-module $\mathscr{M}$ is a $\mathscr{D}_X$-module wich, locally on X, admits a finite presentation as in (6.2).

To a coherent $\mathscr{D}_X$-module $\mathscr{M}$ is naturally associated its characteristic variety

$$\mathrm{char}(\mathscr{M}) \subset T^*X,$$

[1] According to Mikio Sato (personal communication), at the origin of this idea is the mathematician and philosopher of the 17th century, E. W von Tschirnhaus.

a closed $\mathbb{C}^\times$-conic complex analytic variety. A fundamental result of [SKK73] asserts that $\mathrm{char}(\mathscr{M})$ is co-isotropic. A purely algebraic proof has been obtained later by Gabber [Gab81].

If Y is a complex submanifold of X, one defines the induced $\mathscr{D}_Y$-module, $\mathscr{M}_Y$. One says that Y is non-characteristic for $\mathscr{M}$ if f_d is proper on $f_\pi^{-1}\,\mathrm{char}(\mathscr{M})$.

The classical Cauchy-Kowalevska theorem has been extended to $\mathscr{D}$-modules by Kashiwara in his master's thesis.

Theorem 6.1 (see [Kas70]). *Assume that Y is non-characteristic for $\mathscr{M}$. Then $\mathscr{M}_Y$ is $\mathscr{D}_Y$-coherent and one has the natural isomorphism*

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X)|_Y \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y,\mathscr{O}_Y).$$

The inclusion $\subset$ in (6.5) below is also deduced from the Cauchy-Kowalevska theorem, but now in its precised form, as formulated more or less explicitly by Leray [Ler57] after Petrowsky, and more recently by Zerner [Zer71].

Theorem 6.2 (see [KS90, Th. 11.3.3]). *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module and let $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X)$ be the complex of its holomorphic solutions. Then*

$$\mathrm{SS}(F) = \mathrm{char}(\mathscr{M}). \tag{6.5}$$

The inclusion $\supset$ uses the fact that the sheaf $\mathscr{D}_X^\infty$ of differential operators of infinite order is faithfully flat over $\mathscr{D}_X$ (a result of M. Kashiwara) and Sato's formula

$$\mathscr{D}_X^\infty \simeq H^n_\Delta(\Omega^{(0,n)}_{X\times X}),\ \text{with } n = \dim_{\mathbb{C}} X.$$

7 Hyperfunction solutions of hyperbolic systems

From now on, M is a real analytic manifold, say of dimension n, and X is a complexification of M. Recall the sheaf $\mathscr{A}_M$ of real analytic functions on M and the sheaf $\mathscr{B}_M$ of Sato's hyperfunctions on M, defined by

$$\begin{aligned}&\mathscr{A}_M := \mathscr{O}_X \otimes \mathbb{C}_M,\\ &\mathscr{B}_M := \mathrm{R}\mathscr{H}om\,(\mathrm{D}'_X\mathbb{C}_M,\mathscr{O}_X).\end{aligned}$$

Here $\mathrm{D}'_X\mathbb{C}_M = \mathrm{R}\mathscr{H}om\,(\mathbb{C}_M,\mathbb{C}_X) \simeq \mathrm{or}_M\,[n]$. It is proved that $\mathscr{B}_M$ is concentrated in degree 0. Hence

$$\mathscr{B}_M \simeq H^n_M(\mathscr{O}_X) \otimes \mathrm{or}_M\,.$$

Definition 7.1. Let $\mathscr{M}$ be a coherent left $\mathscr{D}_X$-module. One sets

$$\mathrm{char}_M(\mathscr{M}) = T^*M \cap C_{T^*_MX}(\mathrm{char}(\mathscr{M})) \tag{7.1}$$

and call $\mathrm{char}_M(\mathscr{M})$ the *hyperbolic characteristic variety* of $\mathscr{M}$ along M.

Applying Theorems 6.2 and 3.5, we get:

Theorem 7.2. *Let $\mathscr{M}$ be a coherent left $\mathscr{D}_X$-module and set $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)$. Then $\mathrm{SS}(F) \subset \mathrm{char}_M(\mathscr{M})$.*

Corollary 7.3. *Let $\mathscr{M}$ be a coherent left $\mathscr{D}_X$-module and let N be a closed analytic submanifold of M, Y a complexification of N in X.. Assume that $T^*_N M \cap \mathrm{char}_M(\mathscr{M}) \subset T^*_M M$. Then*

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)|_N \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N).$$

Here, $\mathscr{M}_Y$ is the inverse image on Y of $\mathscr{M}$ as a $\mathscr{D}$-module.

In other words, the Cauchy problem for hyperfunctions is well-posed in the hyperbolic directions.

Corollary 7.4. *Let (M, λ, q) be a real analytic G-causal manifold (hence, we assume q real analytic), X a complexification of M and let $\mathscr{M}$ be a coherent left $\mathscr{D}_X$-module. Let $M_0 = q^{-1}(t_0)$ for some $t_0 \in q(M)$, Y a complexification of M_0 in X.. Assume that $\mathrm{char}_M(\mathscr{M}) \cap \lambda \cup \lambda^a \subset T^*_M M$. Then*

$$\mathrm{R}\Gamma(M; \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)) \xrightarrow{\sim} \mathrm{R}\Gamma(M_0; \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_{M_0})).$$

In other words, the Cauchy problem for hyperfunctions is *globally* well-posed.

There also exists a statement (see [JS16]) for M_0 replaced with a higher codimensional submanifold.

Example 7.5. $M = N \times \mathbb{R}$ where N is a compact manifold. Choose $\lambda = T^*_N N \times \{\tau \geq 0\} \subset T^*M$. Let P be a differential operator of order 2 such that

$$(7.2) \qquad \begin{cases} P = \partial_t^2 - R, \\ \sigma_2(R)|_{T^*_M X} \leq 0, \\ R \text{ commutes with } t. \end{cases}$$

Then P is hyperbolic in the codirections $(x, t; \pm dt)$ for all $(x, t) \in M$ and the Cauchy problem

$$(7.3) \qquad \begin{cases} Pf = 0 \\ (f, \partial_t f)|_{t=0} = (h_0, h_1) \end{cases}$$

is globally well-posed for hyperfunctions.

Assume now that $q\colon M \to \mathbb{R}_{>0}$ is a time function and that $N_t = q^{-1}(t)$ is a compact manifold endowed with a Riemannian structure and denote by Δ_t the associated Laplacian. We get a kind of wave equation

$$\partial_t^2 - \Delta_t$$

What happens for $t < 0$ if the diameter of N_t goes to 0?

8 Before the Big Bang

Some mathematicians [MM14] and physicists [Pen12] proposed their own interpretation of the Big Bang. Here, we give a purely mathematical candidate which concerns in fact any phase transition, a phenomena happens in particular when the rank of the projection on M of a smooth Lagrangian submanifold of T^*M changes.

Let us represent the universe as a closed ball in $\mathbb{R}^n$ whose radius grows linearly with the time t. What happens for $t < 0$? If one replaces the spacetime with the constant sheaf supported by it, the sheaf $\mathbf{k}_{\{|x|\leq t\}}$ defined on $t \geq 0$, we need to extend it naturally for $t < 0$. The micro-support of this sheaf at the boundary is the interior conormal. If we extend it naturally for $t < 0$ we get the exterior conormal which is the micro-support of the constant sheaf on the open cone.

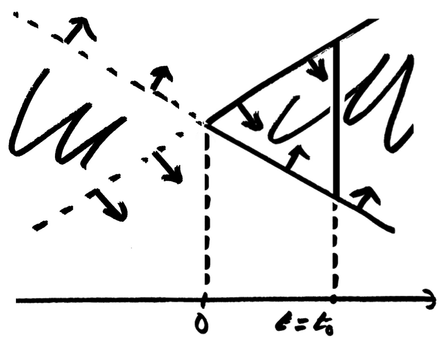

Figure 2: Before the Big Bang

With Guillermou and Kashiwara [GKS12, Exa. 3.10, 3.11], we have constructed a “distinguished triangle” as follows. Set $X = \mathbb{R}^n_x \times \mathbb{R}_t$. We have the isomorphisms

$$\mathrm{D}'_X \mathbf{k}_{\{0\}} \simeq \mathbf{k}_{\{0\}}\,[-n-1], \quad \mathrm{D}'_X \mathbf{k}_{\{|x|\leq -t\}} \simeq \mathbf{k}_{\{|x|<-t\}}$$

The morphism $\mathbf{k}_{\{|x|\leq -t\}} \to \mathbf{k}_{\{0\}}$ induces by duality the morphism

$$\mathbf{k}_{\{0\}} \to \mathbf{k}_{\{|x|<-t\}}[n+1].$$

Composing with $\mathbf{k}_{\{|x|\leq t\}} \to \mathbf{k}_{\{0\}}$, we get the morphism $\mathbf{k}_{\{|x|\leq t\}} \xrightarrow{\psi} \mathbf{k}_{\{|x|<-t\}}\,[n+1]$ hence a distinguished triangle

$$\mathbf{k}_{\{|x|<-t\}}[n] \to K \to \mathbf{k}_{\{|x|\leq t\}} \xrightarrow[\psi]{+1} .$$

One can interpret the sheaf K as the quantization of an Hamiltonian isotopy. Indeed, let $M = \mathbb{R}^n$ and denote by $(x;\xi)$ the homogeneous symplectic coordinates on $T^*\mathbb{R}^n$. Consider the isotopy

$$\varphi_t(x;\xi) = (x - t\frac{\xi}{|\xi|};\xi), \quad t \in I = \mathbb{R}.$$

The micro-support of K outside the zero-section is the smooth Lagrangian manifold, the image of $T^*_{\{0\}}\mathbb{R}^n$ by this

One can modify the Lorentzian case encountered above and replace $\mathbb{R}^n_x$ with a Riemannian manifold (with convexity radius and injectivity radius > 0) using the Hamiltonian isotopy associated with $||\xi||_x$. In particular, one can consider the Euclidian n-sphere $M = \mathbb{S}^n$ $(n \geq 2)$. In this case, the sheaf obtained has a shift which jumps by the dimension at each pole.

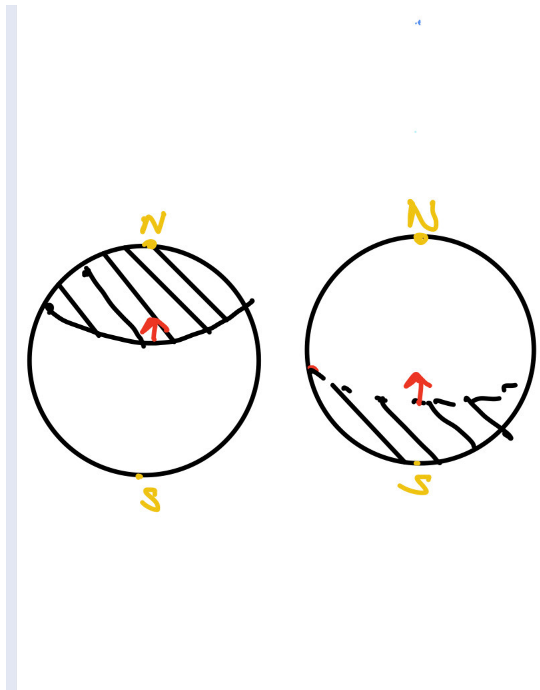

Figure 3: Periodic Big Bang

One finds a periodic Big Bang.

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