



Landau damping in the Kuramoto model

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The Kuramoto model of coupled oscillators [Kuramoto 75,84]

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i), \quad \forall i = 1, \dots, N \quad \theta_i \in \mathbb{T}^1$$

- $\omega_i \in \mathbb{R}$, random with distribution g
 - $K > 0$ coupling parameter
- Heterogeneities vs. (mean-field) Interactions

State-of-art: *Transition to synchrony* [Ermentrout 85, Mirollo & Strogatz 05, Ha et al. 10, Benedetto et al. 15]

- $g = \text{Dirac distribution} \Rightarrow \lim_{t \rightarrow +\infty} \max_{i,j} |\theta_i(t) - \theta_j(t)| = 0, \quad \forall K > 0 \quad \text{Full sync in homogeneous systems}$
- g has bounded support $\Rightarrow \lim_{t \rightarrow +\infty} |\theta_i(t) - \theta_j(t)| = \theta_{ij}, \quad \forall K > |\text{supp}(g)| \quad \text{Locked states in weakly heterogeneous systems}$

Open problems:

- Dynamics in weak coupling regime $K < K_c$?
 - g of unbounded support?
- Continuum limit approximation ($N \rightarrow +\infty$)

Continuum limit / Dynamics of empirical measures: $d\mu_N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{\theta_i(t), \omega_i}$ weak solution of

$$\partial_t f + \partial_\theta (f V[f]) = 0 \quad \text{where} \quad V[f](\theta, \omega) = \omega + K \int_{\mathbb{T}^1 \times \mathbb{R}} \sin(\theta' - \theta) f(d\theta', d\omega') \quad \text{'Kuramoto PDE'}$$

f = measure on $\mathbb{T}^1 \times \mathbb{R}$

Kuramoto PDE and the continuum limit

$$\partial_t f + \partial_\theta (f V[f]) = 0 \quad \text{where} \quad V[f](\theta, \omega) = \omega + K \int_{T^1 \times \mathbb{R}} \sin(\theta' - \theta) f(d\theta', d\omega')$$

Basic features of the Kuramoto PDE: [Lancellotti 05]

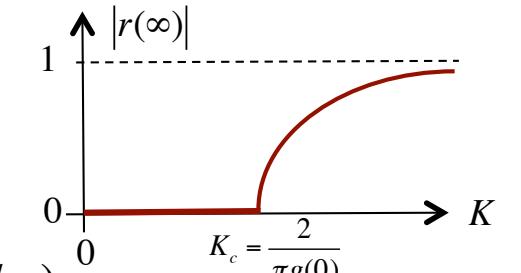
- Cauchy problem well-posed (weak formulation), for every initial probability measure $f(0)$
- $\int_{T^1} f(t, d\theta, d\omega) = g(d\omega), \forall t > 0$ *Invariance of frequency marginal*
- Continuous dependence on initial condition (weak convergence/Bounded Lipschitz distance $d_{BL}(\cdot, \cdot)$)

→ Justification of continuum approximation $\Leftarrow d_{BL}(d\mu_N(t), f(t)) \leq d_{BL}(d\mu_N(0), f(0))e^{Ct}, \forall t > 0$

Phenomenology: (g unimodal and even) [Strogatz 00]

- Homogeneous stat. state $f_{inc}(\theta, \omega) = \frac{g(\omega)}{2\pi}$ is stable for $K < K_c$ $r_{inc} = 0$
- Circle of stable partially locked state emerges at $K = K_c$ $|r_{pls}| > 0$

→ Bifurcation diagram for order parameter $r(t) = \int_{T^1 \times \mathbb{R}} e^{i\theta} f(t, d\theta, d\omega)$



State-of-art on math results: [Ott & Antonsen 08, Martens et al. 09, Strogatz et al. 91, 92, Mirollo 12, Chiba 14, etc]

- Stability criteria (ie. def of K_c)
- Linear analysis (\Rightarrow relaxation behavior depends on regularity of g)
- Nonlinear analysis of homogeneous state stability for $g =$ Gaussian, Cauchy distr.

Main issue: [Mirollo & Strogatz 07, Nordenfelt 15, Dietert 16,]

Continuous spectrum on imaginary axis. Stationary states unstable in L^2 -norm!

Incoherent state stability $f_{inc}(\theta, \omega) = \frac{g(\omega)}{2\pi}$

- $\|u\|_{L^1(\mathbb{R}^+, (1+t)^n)} = \int_{\mathbb{R}^+} (1+t)^n |u(t)| dt \quad n \in \mathbb{N}$
- $\|u\|_{H^n}^2 = \sum_{k_\theta, k_\omega \geq 0, k_\theta + k_\omega \leq n} \left\| \sqrt{1+\omega^2} \partial_\theta^{k_\theta} \partial_\omega^{k_\omega} u \right\|_{L^2(\mathbb{T}^1 \times \mathbb{R})}^2$ *Weighted Sobolev norm* [Faou & Rousset 16]

Lem: $\tau^n |\hat{u}_k(\tau)| \leq \text{Cst} \|u\|_{H^n}, \quad \forall \tau > 0, k \in \mathbb{Z}$
↑
Fourier transform

Theorem: [F, Gérard-Varet & Giacomin 16]

Assume that $\|g\|_{H^n} < +\infty$ and $\|\hat{g}\|_{L^1(\mathbb{R}^+, (1+\tau)^n)} < +\infty$ for $n \geq 4$.

Then, for every $K > 0$ such that $1 - \frac{K}{2} \int_{\mathbb{R}^+} \hat{g}(\tau) e^{\lambda\tau} d\tau \neq 0 \quad \forall \lambda \text{ with } \operatorname{Re}(\lambda) \leq 0$, **Stability condition**

there exists $\varepsilon > 0$ such that for any $f(0)$ with marginal g and $\|f(0) - f_{inc}\|_{H^n} < \varepsilon$,

we have

- $|r(t)| \leq \text{Cst} \cdot t^{-n}, \quad \forall t > 0$ **“algebraic Landau damping”**
- $\exists f_\infty \neq f_{inc}$ such that $\|f(t, \theta + \omega t, \omega) - f_\infty(\theta, \omega)\|_{H^{n-2}} = O(t^{-1}) \quad \left(+ \lim_{t \rightarrow +\infty} f(t) = f_{inc} \right)$ [Dietert 16]

Remarks: • $g =$ Gaussian, Cauchy, bi-Cauchy: ok for any $n \geq 4$.

- Stability condition optimal and $\Leftrightarrow \left[\int_{\mathbb{R}^+} \frac{g(\omega - \sigma) - g(\omega + \sigma)}{\sigma} d\sigma = 0 \Rightarrow K < \frac{2}{\pi g(\omega)} \right] \quad \text{Penrose criterion}$
[Strogatz & Mirollo 91]
- $\Leftrightarrow K < \frac{2}{\pi g(0)}$ when g unimodal and even

- Restriction on $\|f(0) - f_{inc}\|_{H^n}$ justified by occurrence of sub-critical bifurcations but

Proposition: For any $\|f(0)\|_{H^n} < +\infty$, $\exists K_{f(0)} > 0$ s.t. conclusions hold $\forall K < K_{f(0)}$.

- Improved conditions on g & exponential decay, see [Dietert 16]

Perturbation dynamics – Landau damping

$$u(t, \theta, \omega) = f(t, \theta + \omega t, \omega) - f_{inc}(\theta, \omega)$$

- Dynamics in Fourier variables $\hat{u}_\ell(t, \tau) = \int_{T^1 \times \mathbb{R}} u(t, \theta, \omega) e^{-i(\ell\theta + \tau\omega)} d\theta d\omega, \quad \forall \ell \in \mathbb{Z}, \tau \in \mathbb{R}$ Fourier transform


$$\left\{ \begin{array}{l} \partial_t \hat{u}_1(t, \tau) - \frac{K}{2} (\bar{r}(t) \hat{g}(t - \tau) - r(t) \hat{u}_2(t, t + \tau)) = 0 \\ \partial_t \hat{u}_\ell(t, \tau) - \frac{kK}{2} (\bar{r}(t) \hat{u}_{\ell-1}(t, t - \tau) - r(t) \hat{u}_{\ell+1}(t, t + \tau)) = 0, \quad \forall \ell > 1 \end{array} \right.$$

Remark:
 $r(t) = \frac{\hat{u}_1(t, t)}{\hat{u}_1(t, t)}$

- Volterra equation for the order parameter [~Penrose approach to Landau damping in Vlasov equation]

Duhamel's formula $\Rightarrow \bar{r} - \frac{K}{2} (\hat{g} * \bar{r}) = F$ where $F(t) = \hat{u}_1(0, t) + \int_0^t r(s) \hat{u}_2(s, s+t) ds$

 $\bar{r} = H_{\frac{K}{2} \hat{g}} * F$ where $H_G = 1 + \sum_{j=1}^{+\infty} G^{(*j)}$ and $G^{(*1)} = G, \quad G^{(*(j+1))} = G * G^{(*j)}$ (well-defined because $g \in L^1(\mathbb{R})$)

- Polynomial control, stability criterion [Dietert 16]

Young's inequality $\Rightarrow \|H_G * F\|_{L^\infty(\mathbb{R}^+, (1+t)^n)} \leq \|H_G\|_{L^1(\mathbb{R}^+, (1+t)^n)} \|F\|_{L^\infty(\mathbb{R}^+, (1+t)^n)}$

Half-line Gelfand Theorem [Gripenberg, Londen & Staffans 90]

Assume that $\|G\|_{L^1(\mathbb{R}^+, (1+t)^n)} < +\infty$ for some $n \in \mathbb{N}$. Then, whenever $1 - \int_{\mathbb{R}^+} G(t) e^{\lambda t} dt \neq 0, \quad \forall \lambda$ with $\text{Re}(\lambda) \leq 0$, we have $\|H_G\|_{L^1(\mathbb{R}^+, (1+t)^n)} < +\infty$

 Stability condition $\Rightarrow \|(1+t)^n r(t)\|_{L^\infty(\mathbb{R}^+)} \leq \text{Cst} \|(1+t)^n F(t)\|_{L^\infty(\mathbb{R}^+)}$

& $t^n |\hat{u}_1(0, t)| \leq \|f(0) - f_{inc}\|_{H^n} \Rightarrow$ Landau damping for linearised dynamics

- Control of nonlinear terms

Bootstrap argument inspired from proof of Landau damping in HMF model by [Faou & Rousset 16]

Completing the bifurcation diagram: Partially locked states analysis

Partially locked state (PLS) = stat. state f_{pls} with $r_{pls} \neq 0$ (axial rotation sym. \Rightarrow wlog assume $r_{pls} \in \mathbb{R}^+$)

PLS equation: $\partial_\theta ((\omega - Kr_{pls} \sin \theta) f_{pls}) = 0$ [Strogatz oo] $(\dot{\theta} = \omega - Kr_{pls} \sin \theta)$ • $R_\theta f(\theta', \omega) = f(\theta' + \theta, \omega)$

$$\rightarrow f_{pls}(\theta, \omega) = \begin{cases} \left(\alpha(\omega) \delta_{\arcsin(\frac{\omega}{Kr_{pls}})}(\theta) + (1 - \alpha(\omega)) \delta_{\pi - \arcsin(\frac{\omega}{Kr_{pls}})}(\theta) \right) g(\omega) & \text{if } |\omega| \leq Kr_{pls} \\ \frac{\sqrt{\omega^2 - (Kr_{pls})^2}}{2\pi|\omega - Kr_{pls} \sin \theta|} g(\omega) & \text{if } |\omega| > Kr_{pls} \end{cases} \quad (\alpha = \text{arbitrary meas. function } \in [0,1])$$

Singular heterogeneous state!

- existence condition $\Leftrightarrow \int_{T^1 \times \mathbb{R}} e^{i\theta} f_{pls}(d\theta, d\omega) = r_{pls}$
- $g \in C^0$ and even $\Rightarrow f_s = f_{pls}(\alpha=1)$ exists $\forall K > K_c$

Preliminaries on stability

- $g \in L^\infty(\mathbb{R})$ and has analytic cont. in $|\text{Im}(\lambda)| < a$
- $\|u\|_a = \left(\sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}} e^{2a\tau} \left(|u_\ell(\tau)|^2 + |u'_\ell(\tau)|^2 \right) d\tau \right)^{\frac{1}{2}}$ Adapted norm

\Rightarrow weak topology

Stability condition

$$\left\{ \begin{array}{l} \det\left(\text{Id} - \frac{K}{2} M(\lambda, r_s)\right) \neq 0, \forall \lambda \neq 0 \text{ with } \text{Re}(\lambda) \geq 0 \\ \liminf_{\lambda \rightarrow 0} \left| \frac{1}{\lambda} \det\left(\text{Id} - \frac{K}{2} M(\lambda, r_s)\right) \right| > 0 \end{array} \right. \quad (S)$$

- g unimodal and even $\Rightarrow f_s$ unique & (S) holds $\forall K > K_c$

Proposition: [Dietert, F & Gérard-Varet 16]

- $\|f_{pls}\|_a < +\infty$ iff $\alpha=1$ a.e. Stability requirement
- $\|\hat{f}(0)\|_a < +\infty \Rightarrow \|\hat{f}(t)\|_a < +\infty, \forall t > 0$ Well-posedness

$$\bullet M(\lambda, r) = \begin{pmatrix} J_0(\lambda, r) & J_2(\lambda, r) \\ \overline{J_2(\bar{\lambda}, r)} & \overline{J_0(\bar{\lambda}, r)} \end{pmatrix}$$

$$\bullet J_k(\lambda, r) = \int_{\mathbb{R}} \frac{\beta^k(\frac{\omega}{Kr}) g(\omega)}{\lambda + i\omega + Kr\beta(\frac{\omega}{Kr})} d\omega$$

$$\bullet \beta(\omega) = -i\omega + \begin{cases} \sqrt{1 - \omega^2} & \text{if } |\omega| \leq 1 \\ i\omega\sqrt{1 - \omega^2} & \text{if } |\omega| > 1 \end{cases}$$

Asymptotic stability of PLS circle

Theorem: [Dietert, F & Gérard-Varet 16]

- $R_\theta f(\theta', \omega) = f(\theta' + \theta, \omega)$

Assume that (S) holds. Then, $\exists \varepsilon, b > 0$ such that for any $f(0)$ with marginal g and $\|\hat{f}(0) - \hat{f}_s\|_a < \varepsilon$,

$$\exists \theta \in T^1 \text{ such that } \|\hat{f}(t) - R_\theta \hat{f}_s\|_a = O(e^{-bt}) \quad \left(\Rightarrow \lim_{t \rightarrow +\infty} f(t) = R_\theta f_s \text{ weak top.} \right)$$

Remarks:

- $g = \text{Gaussian, Cauchy: ok for } K > K_c$, • $g = \text{Bi-Cauchy: sub-critical bifurcation}$
- Algebraic decay [Dietert 17]

Analysis of dynamics in Fourier variables: $u(t) = \hat{f}(t) - \hat{f}_s \Rightarrow \partial_t u = Lu + Qu$ where $L = L_1 + L_2$

- $(L_1 u)_\ell(\tau) = \ell \left(\partial_\tau u_\ell(\tau) + \frac{K_r}{2} (u_{\ell-1}(\tau) - u_{\ell+1}(\tau)) \right)$
- $(L_2 u)_\ell(\tau) = \frac{K\ell}{2} \left(u_1(0)(\hat{f}_s)_{\ell-1}(\tau) - \bar{u}_1(0)(\hat{f}_s)_{\ell+1}(\tau) \right)$
- $(Qu)_\ell(\tau) = \frac{K\ell}{2} (u_1(0)u_{\ell-1}(\tau) - \bar{u}_1(0)u_{\ell+1}(\tau))$

Complexification: L_2 bounded, finite rank, but not C-linear \Rightarrow $u \mapsto (u, v)$ such that $\mathcal{L}_{1,2}(u, \bar{u}) = (L_{1,2}u, \overline{L_{1,2}u})$

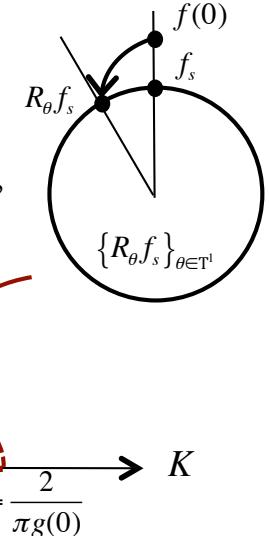
Nonlinearity weak regularity: $\|u\|_{a,0} < +\infty \Rightarrow \|Qu\|_{a,-1} < +\infty$ • $\|u\|_{a,k} = \left(\sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}} e^{2a\tau} \ell^{2k} \left(|u_\ell(\tau)|^2 + |u'_{\ell}(\tau)|^2 \right) d\tau \right)^{\frac{1}{2}}$

Regularising lemma: Let $X \subset Y$ be Hilbert spaces, A be linear op., densely defined and generate semigroup on X and Y .

Assume $\text{Res}(A) \supset \{z : \text{Re}(z) \geq -\gamma\}$ for some $\gamma \geq 0$ and $\sup_{y \in \mathbb{R}} \|((-y + iy)\text{Id} - A)^{-1}\|_{Y \rightarrow X} < +\infty$. • $\|w\|_{H,\gamma} = \left(\int_{\mathbb{R}^+} e^{2\gamma t} \|w(t)\|_H^2 dt \right)^{\frac{1}{2}}$

Assume $\|G\|_{Y,\gamma} < +\infty$. Then, the unique mild solution of $\frac{dw}{dt} = Aw + G$ with $\|w(0)\|_X < \infty$ satisfies

- $w(t) \in X$ for a.e. $t > 0$
- $\|w\|_{X,\gamma} \leq C (\|w(0)\|_X + \|G\|_{Y,\gamma})$ for some $C > 0$



Regularising lemma: Let $X \subset Y$ be Hilbert spaces, A be linear op., densely defined and generate semigroup on X and Y .

Assume $\text{Res}(A) \supset \{z : \text{Re}(z) \geq -\gamma\}$ for some $\gamma \geq 0$ and $\sup_{y \in \mathbb{R}} \|((-\gamma + iy)\text{Id} - A)^{-1}\|_{Y \rightarrow X} < +\infty$. • $\|w\|_{H,\gamma} = \left(\int_{\mathbb{R}^+} e^{2\gamma t} \|w(t)\|_H^2 dt \right)^{\frac{1}{2}}$

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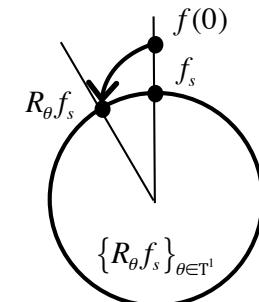
Proposition:

$\mathcal{L}_1 + \mathcal{L}_2$ has the following properties on $\|(u,v)\|_{a,k} < +\infty$ ($k = 0,1$)

- it generates a C^0 -semigroup
- if (S) holds, then $\exists b > 0$ such that $\text{Spec}(\mathcal{L}_1 + \mathcal{L}_2) \cap \text{Re}(\lambda) \geq -b = \{0\}$
- $\sup_{y \in \mathbb{R}} \|((-\gamma + iy)\text{Id} - (\mathcal{L}_1 + \mathcal{L}_2))^{-1}\|_{X_{a,0}^2 \rightarrow X_{a,-1}^2} < +\infty \quad \forall x > -b, x \neq 0$

Additional arguments

- getting rid of the angle coordinate (= direction associated with 0 eigenvalue) $\Rightarrow \hat{f} = \hat{R}_\theta(\hat{f}_s + u)$ in neigbh. of PLS circle
- Forcing $G \Rightarrow$ Nonlinear term Qu
- Control of $\|u\|_{X_{a,0},b} \Rightarrow$ Control of $\sup_{t>0} e^{bt} \|u(t)\|_{X_{a,0}}$



References

- B.F., D. Gérard-Varet and G. Giacomin, Ann. Henri Poincaré 17 (2016) 1793-1823
- H. Dietert, B.F. and D. Gérard-Varet , arXiv 1606.04470